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ON COUNTABLE STABLE STRUCTURES WHICH ARE HOMOGENEOUS FOR A FINITE RELATIONAL LANGUAGE

BY

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Dedicated to the memory of Abraham Robinson on the tenth anniversary of his death

ABSTRACT

Let L be a finite relational language and $\boldsymbol{H}(L)$ denote the class of all countable structures which are stable and homogeneous for L in the sense of Fraissé. By convention countable includes finite and any finite structure is stable. A rank function $r:\boldsymbol{H}(L)\rightarrow\omega$ is introduced and also a notion of dimension for structures in $\boldsymbol{H}(L)$. A canonical way of shrinking structures is defined which reduces their dimensions. The main result of the paper is that any $M\in\boldsymbol{H}(L)$ can be shrunk to $M'\in\boldsymbol{H}(L)$, $M'\subseteq M$, such that $|M'|$ is bounded in terms of $r(M)$, and the isomorphism type of M over M' is uniquely determined by the dimensions of M. For $r<\omega$ we deduce that $\boldsymbol{H}(L,r)$, the class of all $M\in\boldsymbol{H}(L)$ with $r(M)\leqq r$, is the union of a finite number of classes within each of which the isomorphism type of a structure is completely determined by its dimensions.

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Introduction

Throughout the paper "structure" means a countable possibly finite structure for a relational language containing equality, and "homogeneous" means that any isomorphism between finite substructures extends to an automorphism of the whole structure.

Let the class of stable structures homogeneous for the language L be denoted $\boldsymbol{H}(L)$, where stable is in the sense of Shelah [12] and finite structures are automatically stable. If $M \in \boldsymbol{H}(L)$ it is easy to see that M is $\aleph_0$-stable and that the Morley rank of M is finite.

Until further notice let L be the language consisting of one binary relation symbol R. To motivate our work we describe some results about this special case. The first theorem classifying some homogeneous structures was that of Gardiner [5] describing finite homogeneous graphs, where "graph" means symmetric, irreflexive L-structure. If G, H are graphs let $G \times H$ be the usual cartesian product, $G[H]$ be the wreath product obtained by replacing each vertex of G by a copy of H, and $\bar{G}$ be the complement of G, i.e. the graph with the same universe and

$$R^{\bar{G}} = G^2 - (R^G \cup \{\langle g, g\rangle : g \in G\}).$$

Let K_n denote the complete graph on n vertices and P denote the graph whose diagram is a pentagon. Let $\boldsymbol{S}$ denote the class of finite homogeneous graphs. Gardiner showed that $G \in \boldsymbol{S}$ iff G is isomorphic to one of: P, $K_3 \times K_3$, $\bar{K}_m[K_n]$, and $K_m[\bar{K}_n]$ $(1 \leqq m,\ n < \omega)$.

The author [6] has extended Gardiner's work in the following way. Let $\boldsymbol{A}$ denote the class of finite homogeneous antisymmetric digraphs, i.e. the class of finite homogeneous irreflexive L-structures satisfying $\forall x\, \forall y\,(xRy \rightarrow \neg\, yRx)$. Let C, $D \in \boldsymbol{A}$ denote the 3-cycle, 4-cycle respectively. There exists $H_0 \in \boldsymbol{A}$ of cardinality 8 such that $M \in A$ iff M is isomorphic to one of: D, H_0, $\bar{K}_n$, $\bar{K}_n[C]$, and $C[\bar{K}_n]$ $(1 \leqq n < \omega)$. Let $\boldsymbol{I}$ denote the class of finite homogeneous irreflexive L-structures. There are $H_1, H_2 \in \boldsymbol{I}$ of cardinalities 8 and 12 respectively such that $M \in \boldsymbol{I}$ iff either M or $\bar{M}$ is isomorphic to a digraph having one of the following forms: H_1, H_2, S, $K_n[A]$, $A[K_n]$, $C[S]$, and $S[C]$ $(1 \leqq n < \omega,\ S \in \boldsymbol{S}$, $A \in \boldsymbol{A})$.

At this point we are close to having characterized the finite members of $\boldsymbol{H}(L)$. Let $\boldsymbol{R}$ denote the class of finite homogeneous reflexive L-structures. The characterization of $\boldsymbol{R}$ is obtained from that of $\boldsymbol{I}$ by replacing each irreflexive structure by the corresponding reflexive one. If $M \in \boldsymbol{H}(L) - (\boldsymbol{I} \cup \boldsymbol{R})$ is finite then there exist $M_0 \in \boldsymbol{I}$ and $M_1 \in \boldsymbol{R}$ such that M can be obtained from the disjoint union $M_0 \dot{\cup} M_1$ by adding some edges between M_0 and M_1. The possibilities for M_0 and M_1 are known from above. It remains to analyze the ways in which M_0 and M_1 can be linked in M.

From [9] which characterizes the infinite homogeneous graphs it is clear that the infinite stable homogeneous graphs are just the obvious limits of finite ones, i.e. $\bar{K}_m[K_n]$, $K_m[\bar{K}_n]$ with $1 \leqq m, n \leqq \omega$ and at least one of $m, n = \omega$. Also, [3] yields a general method of reaching the same conclusion. Call $M_0 \subseteq M$ a *weakly homogeneous substructure* of $M \in \boldsymbol{H}(L)$ if two tuples from M_0 realize the same type in M_0 iff they realize the same type in M. In fact, as the referee observed, if $M \in \boldsymbol{H}(L)$, then $M_0 \subseteq M$ is a weakly homogeneous substructure iff $M_0 \in \boldsymbol{H}(L)$. If $M \in \boldsymbol{H}(L)$ is infinite then M is both $\aleph_0$-stable and $\aleph_0$-categorical. Corollary 7.4 of [3] says that for any $\aleph_0$-stable, $\aleph_0$-categorical structure M and any sentence ϕ there is a finite weakly homogeneous $M_0 \subseteq M$ such that $M_0 \models \phi$ iff $M \models \phi$. It follows easily that not only are the infinite stable graphs the obvious limits of ones in $\boldsymbol{S}$ but the corresponding results are true for $\boldsymbol{A}$, $\boldsymbol{I}$, and $\boldsymbol{R}$.

There is no doubt that the classification of structures in $\boldsymbol{H}(L)$ can be pushed to a conclusion using the method of Gardiner which was also employed in [6]. The only obstacle is the sheer number of cases to be examined. When the classification is complete we expect to have expressed $\boldsymbol{H}(L)$ as the union of a finite number of easily describable families just as we can write

$$\boldsymbol{S} = \{P\} \cup \{K_3 \times K_3\} \cup \{\bar{K}_m[K_n] : 1 \leqq m, n < \omega\} \cup \{K_m[\bar{K}_n] : 1 \leqq m, n < \omega\}.$$

We now drop the convention that L is the language with one binary relation. Based on our knowledge of that special case we conjecture that for any finite relational language L the members of $\boldsymbol{H}(L)$ can be classified into a finite number of easily describable families. In this paper we formulate two precise versions of this conjecture, 0.2 and 0.3 below, and reduce them to another conjecture 0.1 about rank.

Since the latter conjecture is easier to state we make a brief digression in order to formulate it before saying more about the former conjectures. For a structure M we define the *rank of* M denoted $r(M)$ to be $\sup\{\mathrm{CR}(p, 2) : p \in S(\emptyset)\}$ where $\mathrm{CR}(p, \lambda)$ is the complete rank of Shelah [12, p. 55]. This rank notion is particularly well-suited to the study of finite structures for which other rank notions, e.g. that of Morley [10], are too coarse. For $M \in \boldsymbol{H}(L)$, $r(M)$ is always defined and finite. Our conjecture about rank which we do not attempt to prove is:

CONJECTURE 0.1. *If L is a finite relational language there exists $r < \omega$ such that $r(M) < r$ for all $M \in \boldsymbol{H}(L)$.*

Shelah and the author have proved 0.1 in the case where L contains at most unary and binary relation symbols. That proof appears in [8]. More recently the full conjecture has been confirmed by Cherlin and the author [2]. The latter rests heavily on the theory of permutation groups and uses the classification of finite simple groups. Returning to the conjecture concerned with classifying members of $\boldsymbol{H}(L)$ we introduce a notion of dimension for structures in $\boldsymbol{H}(L)$. The key concept is that of a nice pair. Let $\langle \phi_0(\bar{x}), \phi_1(\bar{x})\rangle$ be a pair of quantifier-free formulas of L with $\bar{x} = \langle x_0, x_1, x_2, x_3\rangle$. If $M \in \boldsymbol{H}(L)$ we say that $\boldsymbol{\phi} = \langle \phi_0, \phi_1\rangle$ is a *nice pair for* M if the following five conditions are satisfied:

(i) ϕ_0, ϕ_1 define equivalence relations E_0, E_1 on the same subset

$$D = \{a \in M^2 : M \models \phi_i(a, a)\} \qquad (i < 2)$$

of M^2 and $E_1 \subseteq E_0$,

(ii) $\mathscr{F} = \{C/E_1 : C \in D/E_0\}$ is a family of mutually indiscernible sets,

(iii) $\mathscr{F}$ is permuted transitively by aut(M),

(iv) for all $a \in M$ there is a least subset $\mathrm{cl}^{\boldsymbol{\phi}}(\{a\}) \subseteq \bigcup \mathscr{F}$ such that $|\mathrm{cl}^{\boldsymbol{\phi}}(\{a\})| < \aleph_0$, and for all $I \in \mathscr{F}$ $|\mathrm{cl}^{\boldsymbol{\phi}}(\{a\}) \cap I| < |I|/8$ and $I - \mathrm{cl}^{\boldsymbol{\phi}}(\{a\})$ is strongly indiscernible over $\{a\} \cup \mathrm{cl}^{\boldsymbol{\phi}}(\{a\}) \cup ((\bigcup \mathscr{F}) - I)$,

(v) for all $I \in \mathscr{F}$ and $J \subseteq I$, J is strongly indiscernible over A, where

$$A = \{a \in M : \mathrm{cl}^{\boldsymbol{\phi}}(\{a\}) \cap J = \emptyset\} \cup ((\bigcup \mathscr{F}) - J).$$

When $\boldsymbol{\phi}$ is a nice pair for M we write $d_M(\boldsymbol{\phi}) = n$ where n is the common cardinality of the sets I in $\mathscr{F}$, and we say that M has *$\boldsymbol{\phi}$-dimension n*. Otherwise $d_M(\boldsymbol{\phi})$ is undefined and M has no $\boldsymbol{\phi}$-dimension.

Before explaining the significance of nice pairs we give an illustration which shows why in general the equivalence relations E_0, E_1 should be on a subset of M^2 rather than on a subset of M. Let $L = \{R, P\}$ where R is a binary relation symbol and P is quaternary. Let S_n $(1 \leqq n \leqq \omega)$ be the L-structure whose restriction to $\{R\}$ is $\overline{K_n \times K_n}$ and such that

$$P^{S_n} = \{\langle a_i : i < 4\rangle \in S_n^4 : (\forall i, j < 4)(a_i R^{S_n} a_j \equiv \{i, j\} \in \{\{0,1\},\{2,3\}\}) \\ \& \ (\exists a \in S_n)(\forall i < 4)(a R^{S_n} a_i)\}.$$

A diagram of S_6 is shown in Fig. 1. Two vertices are R-related if they are distinct and on the same horizontal or vertical line. The vertices labelled a_0, a_1, a_2, a_3 are a typical quadruple satisfying P.

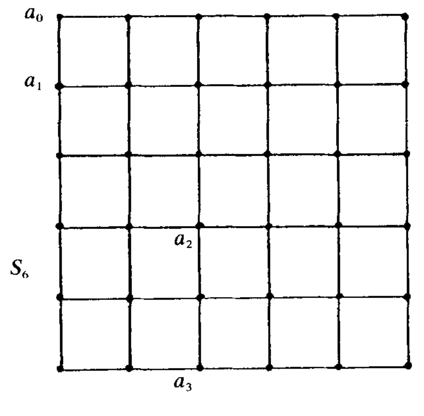

Fig. 1.

Obviously P is definable from R. We include P in L to ensure that $S_n \in \boldsymbol{H}(L)$ ($1 \leqq n < \omega$). When $L = \{R\}$ then only S_1, S_2, S_3 are homogeneous for L.

We say that $\{x, y\}$ *determines* λ if $x, y \in S_n$ are distinct, λ is one of the horizontal or vertical lines of S_n, and x, y lie on λ. Let $\bar{x} = \langle x_0, x_1, x_2, x_3 \rangle$. Let $\phi_0(\bar{x})$ be a quantifier-free formula saying $\{x_0, x_1\}$, $\{x_2, x_3\}$ determine lines which are either the same or parallel, and $\phi_1(\bar{x})$ be a quantifier-free formula saying $\{x_0, x_1\}$, $\{x_2, x_3\}$ determine the same line. Then $\boldsymbol{\phi} = \langle \phi_0, \phi_1 \rangle$ is a nice pair for S_n ($8 \leqq n \leqq \omega$).

The family $\mathcal{F} = \{C/E_1 : C \in D/E_0\}$ of mutually indiscernible sets defined by $\boldsymbol{\phi}$ has two members of cardinality n. Intuitively, one member of $\mathcal{F}$ consists of the horizontal lines and the other of the vertical lines. For each $a \in S_n$, $\mathrm{cl}^{\boldsymbol{\phi}}(\{a\})$ is the doubleton whose members are the horizontal and vertical lines through a. By inspection $\boldsymbol{\phi}$ satisfies all the requirements for a nice pair. Thus $d_{S_n}(\boldsymbol{\phi}) = n$ ($8 \leqq n \leqq \omega$).

Returning to the general discussion, the importance of nice pairs is that they allow us to "shrink" structures. Suppose $M \in \boldsymbol{H}(L)$, $\boldsymbol{\phi}$ is a nice pair for M, and $m < d_M(\boldsymbol{\phi})$. Define $M' \subseteq M$ as follows. Let $\mathcal{F}$ be as in the definition of nice pair above. Let $B \subseteq \bigcup \mathcal{F}$ be any subset satisfying $|B \cap (C/E_1)| = m$ for all $C \in D/E_0$ and $M' \subseteq M$ be the substructure with universe $\{a \in M : \mathrm{cl}^{\boldsymbol{\phi}}(\{a\}) \subseteq B\}$. We assume m is large enough for M' to be nonempty. Since $\mathcal{F}$ is a family of mutually indiscernible sets the isomorphism type of M' depends only on m. Further $M' \in \boldsymbol{H}(L)$ and provided m is not too small we shall have $d_{M'}(\boldsymbol{\phi}) = m$. Call the process that generates M' from M a *one-step shrinking* of M, and denote M' by $M(\boldsymbol{\phi}, m)$. We call m the *target dimension*.

Let $\boldsymbol{\phi}_0$, $\boldsymbol{\phi}_1$ be nice pairs for M and $\mathcal{F}_0$, $\mathcal{F}_1$ be the corresponding families of mutually indiscernible sets. We call ϕ_0, ϕ_1 *equivalent for* M if there is 0-definable bijection between $\bigcup \mathcal{F}_0$ and $\bigcup \mathcal{F}_1$. Clearly, if ϕ_0, ϕ_1 are equivalent for M then $d_M(\phi_0) = d_M(\phi_1)$.

Let $\Phi(M)$ denote the class of all nice pairs for M and $\approx_M$ be the relation of equivalence for M on $\Phi(M)$. $\Phi(M)$ is essentially finite, because up to logical equivalence there are only a finite number of quantifier-free formulas $\phi(\bar{x})$ of L with $\bar{x} = \langle x_0, x_1, x_2, x_3 \rangle$. A one-step shrinking of M to $M' = M(\phi, m)$ is called *normal* if $\Phi(M') = \Phi(M)$, $\approx_{M'} = \approx_M$, $d_{M'}(\phi) = m$, and $d_{M'}(\psi) = d_M(\psi)$ for all $\psi \in \Phi(M)$ such that ϕ, ψ are not equivalent.

We say that $N \in \boldsymbol{H}(L)$ *is obtained by shrinking* $M \in \boldsymbol{H}(L)$ if N can be obtained from M by a finite sequence of normal one-step shrinkings.

We say that $N \in \boldsymbol{H}(L)$ is obtained from $M \in \boldsymbol{H}(L)$ by a *smooth shrinking* if for every $d : \Phi(M) \to \omega$ such that $d_N \leqq d \leqq d_M$ and $d \notin \{d_M, d_N\}$ there exists $M' \in \boldsymbol{H}(L)$ such that $d_{M'} = d$, M' is obtained by shrinking M, and N is obtained by shrinking M'.

It turns out that one-step shrinkings are normal provided the target dimension is large enough compared with the Gödel number of L. Also a shrinking will be smooth provided the target dimension in each one-step shrinking that is part of it is large enough compared with the Gödel number of L.

Above we conjectured that the members of $\boldsymbol{H}(L)$ can be classified into a finite number of easily describable families. A precise version of that conjecture is

CONJECTURE 0.2. *If L is a finite relational language there exists $s < \omega$ such that for every $M \in \boldsymbol{H}(L)$ there exists $M' \in \boldsymbol{H}(L)$ such that $|M'| \leqq s$ and either $M' = M$, or*

(i) *M' is obtained from M by a smooth shrinking, and*

(ii) *if $N \in \boldsymbol{H}(L)$, $d_N = d_M$, and M' is obtained by shrinking N, then there is an isomorphism $\alpha : M \to N$ such that $M' \subseteq \mathrm{fix}(\alpha)$.*

We could have proceeded in a simpler fashion by defining a shrinking to be any sequence of one-step shrinkings and conjecturing that every $M \in \boldsymbol{H}(L)$ can be shrunk to $M' \in \boldsymbol{H}(L)$ of bounded cardinality such that the isomorphism type of M over M' can be recovered from $\langle M', d_M \rangle$. We prefer the version of 0.2 given because at the end of the paper we shall be able to see that it is equivalent to 0.1. The simpler alternative is implied by 0.1 but we are not sure whether it is equivalent.

Another conjecture which turns out to be equivalent to 0.1 is

CONJECTURE 0.3. *If L is a finite relational language there exists a finite* *subclass $\boldsymbol{H}_0 \subseteq \boldsymbol{H}(L)$ such that every $M \in \boldsymbol{H}(L)$ is isomorphic either to a member of $\boldsymbol{H}_0$ or to a structure obtained by shrinking a member of $\boldsymbol{H}_0$.*

0.3 implies 0.1 because $r(M)$ never increases when we shrink M.

Let $\boldsymbol{H}(L, r)$ denote $\{M \in \boldsymbol{H}(L) : r(M) \leqq r\}$. The main results of the paper are that 0.2 and 0.3 are true if for $\boldsymbol{H}(L)$ we substitute $\boldsymbol{H}(L, r)$ where $r < \omega$ is arbitrary. These results are Theorems 15.1 and 15.3. It follows immediately that 0.1 implies 0.2 and 0.3 because if 0.1 holds then $\boldsymbol{H}(L) = \boldsymbol{H}(L, r)$ for some $r < \omega$ depending on L. It was noted above that 0.3 implies 0.1. In §15 we shall show that 0.2 also implies 0.1. The main observation needed for this is 15.2 which says that, if $\langle M_i : i < \omega \rangle$ is a chain in $\boldsymbol{H}(L)$ such that M_i is obtained from M_{i+1} by a one-step shrinking for each $i < \omega$, then $\bigcup\{M_i : i < \omega\} \in \boldsymbol{H}(L)$.

The rest of the paper is organized into two chapters. The first chapter contains the elements of stability theory for the homogeneous structures we are concerned with. We find it convenient to throw away the language and to look at the corresponding permutation groups instead of at relational structures. This approach is explained in §2. Whenever we need it the language can be recovered. We define $a(M)$ the arity of the structure, i.e. the least $n < \omega$ such that M is homogeneous for some language all of whose relations have arity $\leqq n$. We define $l(M)$ to be the Gödel number of the canonical language for M, i.e. the language whose i-ary relation symbols are in one-one correspondence with the i-types in M for all $i \leqq a(M)$. We also define a notion of homogeneous substructure which is stronger than the notion used in [3].

In §3 we mention some standard results about prime models which will be useful.

In §4 we consider a rank $R(A, \Delta, 2)$ adapted from [12, p. 21] where $A \subseteq M^*$, M^* being an extension by definitions of M, and Δ is a set of formulas. We show that $R(A, \Delta, 2)$ can be bounded in terms of $l(M)$, $r(M)$, the complexity of A, and the complexity of Δ. We also prove that for $k < \omega$, $r(M^k)$ can be bounded in terms of $l(M)$, $r(M)$, and k. This means that results proved for M automatically carry over to cartesian powers.

In §5 we consider the effect on M of naming an element in M^*. Again M^* is an extension by definitions of M so that any particular element of M^* can be seen as an equivalence class of a 0-definable equivalence relation E with $\mathrm{fld}(E) \subseteq M^k$ for some $k < \omega$. Let M' be obtained from M by naming a single E-class. It will be shown that $l(M')$ and $r(M')$ can be bounded in terms of $l(M)$, $r(M)$, and k.

In §6 we quote a result of Shelah bounding the number of types over a finite set. This is applied in §7 where we develop the theory of indiscernible sets and also in §8 where we show that if $H \subseteq M \in \boldsymbol{H}(L)$ is strongly minimal then H is disintegrated in the sense of Zilber [14]. That completes Chapter I.

In Chapter II we develop the theory of $\boldsymbol{H}(L, r)$ where $\boldsymbol{H}(L, r)$ is the class of all stable M homogeneous for the finite relational language L such that $r(M) \leqq r$ and $r < \omega$. Let $\boldsymbol{H}$ denote the union of all the classes $\boldsymbol{H}(L, r)$.

In §9 we quote the Coordinatization Theorem from [3, 3.1]. Applying the Compactness Theorem we obtain what we call the Trichotomy Theorem (9.4): if $M \in \boldsymbol{H}$ and N is a transitive quotient structure of M then either $|N|$ is small compared with $l(M)$ and $r(M)$, or there is a nontrivial equivalence relation 0-definable on N, or N is perfect. The notions of quotient structure and perfect structure will be defined later; "transitive" means there is only one 1-type over $\varnothing$. A good example of a perfect structure is S_n defined above.

One of the notions defined in §7 is that of the closure $\mathrm{cl}^I(A)$ of a finite set $A \subseteq M$ in an indiscernible set I; this is the least $J \subseteq I$ if any such that J is finite, $2|J| < |I|$, and $I - J$ is indiscernible over $J \cup A$. In §10 we prove the following result about indiscernible sets. If $M \in \boldsymbol{H}$, $I \subseteq M^*$ is a 0-definable indiscernible set in an extension by definitions of M, $J \subseteq I$,

$$A = \{a \in M : J \cap \mathrm{cl}^I(\{a\}) = \varnothing\},$$

and $|I|$ is large enough compared with $l(M)$, $r(M)$, and $[I]$ then J is strongly indiscernible over A. Here $[I]$ denotes the length of a tuple from M representing a member of I.

In §11 we introduce the notion of nice family of indiscernible sets. Using the notation from the discussion of nice pairs above a typical example of a nice family is $\mathscr{F} = \{C/E_1 : C \in D/E_0\}$ associated with a nice pair $\boldsymbol{\phi} = \langle \phi_0, \phi_1 \rangle$. If $M \in \boldsymbol{H}$ and M is large compared with $l(M)$ and $r(M)$ then M can be analysed in terms of a sequence of nice families the length of the sequence being bounded in terms of $l(M)$.

In §12 we discuss shrinking a structure with respect to a nice family of indiscernible sets. The nice families are essentially equivalent to the nice pairs of formulas mentioned above. In §13 and §14 we show that under suitable conditions the shrinking process is uniquely reversible, i.e. knowing the dimensions of the original structure we can recover the original structure from the shrunken one.

Finally in §15 we draw together the results from §11–§14 and prove the main results of the paper already mentioned above.

I would like to thank Jim Loveys for his careful reading of the original typescript, Gregory Cherlin for suggesting use of the Compactness Theorem in §8, and also the Institute for Advanced Studies of the Hebrew University of Jerusalem, where I was a Fellow while most of this work was done.

I owe a considerable debt to the referee who suggested many improvements in the exposition. The referee's influence is greatest in §7. The present proof of 7.1 was suggested by the referee, who also made a substantial improvement to 7.3.

CHAPTER I — PRELIMINARIES

1. Notation and terminology

Relational structures will usually be denoted by M, N possibly with subscripts and superscripts. We shall not distinguish notationally between a structure and its universe. We use $|M|$ to denote the cardinality of M and $L(M)$ to denote the language of M.

For $k<\omega$ let M^k denote the structure with universe M^k whose given relations are all those relations on M^k which are 0-definable (i.e. definable without parameters) in M. Note that M can be identified with the diagonal of M^k if we wish. If M is homogeneous for some finite language, then so is M^k.

An *extension by definitions* of M is a structure obtained as follows. Let $k<\omega$ and $\mathscr{E}$ be a 0-definable equivalence relation with $\mathrm{fld}(\mathscr{E})\subseteq M^k$. Let M^* be the structure with universe $M\dot{\cup}(M^k/\mathscr{E})$, let the relation symbols of M have the same interpretation in M^* as in M, and let M^* have a new $(k+1)$-ary relation symbol R whose interpretation is

$$\{\langle a_0,\dots,a_{k-1},b\rangle : a_0,\dots,a_{k-1}\in M,\ b=\langle a_0,\dots,a_{k-1}\rangle/E\}.$$

This process of adjoining new elements can be repeated a finite number of times. If $|M|>1$, any structure obtained in a finite number of steps will be a 0-definable substructure of one obtained in one step and M may be held fixed. Thus where it is convenient we can always suppose that an extension by definitions is a one-step one.

Throughout the paper M^* denotes an extension by definitions of M which is sufficiently comprehensive to contain all the imaginary elements of M (see [12, III, §6]) that we need in the particular context.

By an M^k*-formula* we mean a formula $\phi(\bar{x};\bar{y})$ in the language of M with $\bar{x}=\langle x_0,\dots,x_{k-1}\rangle$, where $x_0,\dots,x_{k-1}$ are the first k variables. A set Δ of M^k-formulas is a *basis for the* M^k*-formulas* if every M^k-formula $\psi(\bar{x};\bar{z})$ can be expressed as a boolean combination of formulas $\phi(\bar{x};\bar{z})$ each of which is obtained from some $\phi(\bar{x};\bar{y}) \in \Delta$ by substituting a member of $\bar{z}$ for each member of $\bar{y}$.

By a **-formula* we mean one of the form $\phi(x;\bar{y})$ where x runs through M^* and the variables $\bar{y}$ are restricted to M.

If Δ is a finite set of M^k-formulas ($k<\omega$) or *-formulas $\phi(\bar{x};\bar{y})$, $[\Delta]$ denotes $\sup\{l(\bar{y}): \phi(\bar{x};\bar{y}) \in \Delta\}$, the maximum number of parameters.

If $a \in M^*$, then $[a]$ denotes the *complexity of a in M^** which is defined to be the least $k<\omega$ such that for some *-formula $\phi(x;\bar{y})$ with $l(\bar{y})=k$ there exists $\bar{b} \in M$ such that a is the unique solution in M^* of $\phi(x;\bar{b})$.

For $A \subseteq M^*$, $[A]$ denotes the complexity of A, i.e. the least $k<\omega$ such that for some 0-definable equivalence relation E with $\mathrm{fld}(E) \subseteq M^k$ there is a *-formula $\phi(x;\bar{y})$ defining a bijection between A and some subset of M^k/E.

Let $\boldsymbol{C}$, $\boldsymbol{C}(L)$ denote the classes of all structures, L-structures respectively, which are either finite or $\aleph_0$-categorical. Let $\boldsymbol{H}$ denote the class of all structures which are either finite or stable, and homogeneous for some finite language.

When $M \in \boldsymbol{H}$, by a *basic subset of M^k* we mean one defined by an atomic formula or by the negation of an atomic formula of $L(M)$.

If $\phi(\bar{x};\bar{y})$ is an M^k-formula and $\bar{b} \in M$, then $\phi(M;\bar{b})$ denotes the solution set of $\phi(\bar{x};\bar{b})$ in M^k. If $\phi(x;\bar{y})$ is a *-formula and $\bar{b} \in M$ then both $\phi(M;\bar{b})$ and $\phi(M^*;\bar{b})$ denote the solution set of $\phi(x;\bar{b})$ in M^*. We use the notation $\phi(M^*;\bar{b})$ if we wish to emphasise that the set may not be a subset of any M^k, $k<\omega$.

If $k<\omega$ and $A \subseteq M^*$ let $s_k(A)$ denote the set of all complete k-types of M over A. If A is finite then so is $s_k(A)$ and we let $S_k(A)$ denote the set of solution sets in M^k of the types in $s_k(A)$. We shall speak loosely of elements of $S_k(A)$ as types but will denote them by capital letters to distinguish them from elements of $s_k(A)$. By $s^*(A)$ we denote the set of 1-types of M^* over A. If A is finite, then $S^*(A)$ denotes the set of solution sets in M^* of the types in $s^*(A)$. Types are to be complete unless the context requires otherwise. However, in §4 this convention will be suspended.

If $\phi(\bar{x};\bar{y})$ is a formula and $B \subseteq M$ then by a *B-instance* of $\phi(\bar{x};\bar{y})$ we mean $\phi(\bar{x};\bar{b})$ where $\bar{b} \in B$. A subset of M^k ($k<\omega$) or M^* is *B-definable* if it is definable by a B-instance of some formula. The subset is *n-definable* if it is B-definable, for some $B \subseteq M$ with $|B| \leqq n$. In particular, the subset is *0-definable* if it is definable without parameters.

By $\mathscr{E}(M^k)$ we denote the set of all equivalence relations $E \subseteq M^k \times M^k$ which are 0-definable in M; similarly $\mathscr{E}(M^*)$ denotes the set of all 0-definable equivalence relations E with $\mathrm{fld}(E) \subseteq M^*$.

By aut(M) we denote the automorphism group of M. If $\alpha \in \mathrm{aut}(M)$ we regard α as being automatically extended to M^* in the obvious way. If $A, B \subseteq M^*$, then by $\mathrm{aut}_M(A, B)$ we denote $\{\pi \in \mathrm{perm}(A) : (\exists \alpha \in \mathrm{aut}(M))[\pi = \alpha \upharpoonright A \ \& \ B \subseteq \mathrm{fix}(\alpha)]\}$. For any mapping α by fix(α) we mean $\{a \in \mathrm{dom}(\alpha) : \alpha(a) = a\}$.

Let M, N be structures for the relational language L with $M \subseteq N$. We call M a *full substructure of* N if for all k, $1 \leqq k < \omega$, and 0-definable $R \subseteq N^k$ the restriction $R \upharpoonright M^k$ is 0-definable in M. Thus, if $N \in \boldsymbol{H}(L)$, any substructure of N is a full substructure of N. We call M a *weakly homogeneous substructure* of N if for any $\bar{a}, \bar{b} \in M$, $\mathrm{tp}_M(\bar{a}) = \mathrm{tp}_M(\bar{b})$ if and only if $\mathrm{tp}_N(\bar{a}) = \mathrm{tp}_N(\bar{b})$. We call M a *homogeneous substructure* of N if it is full and if for all $\bar{a}, \bar{b} \in M$ with $\mathrm{tp}_N(\bar{a}) = \mathrm{tp}_N(\bar{b})$ there exists $\alpha \in \mathrm{aut}(N)$ such that $\alpha(\bar{a}) = \bar{b}$ and $\alpha(M) = M$. Notice that for substructures homogeneous implies weakly homogeneous. Also if $M \subseteq N$ is weakly homogeneous and $M \in \boldsymbol{C}$ then M is a full substructure of N. For each notion of substructure we have the corresponding notion of extension, e.g. $N \supseteq M$ is called a *homogeneous extension* of M if M is a homogeneous substructure of N.

Let L be a finite relational language and M, N be L-structures which are homogeneous for L. If $M \subseteq N$, then M is a weakly homogeneous substructure of N. Also if $M_0 \subseteq M_1 \subseteq \dots$ is any chain of L-structures each homogeneous for L, then $M = \bigcup\{M_i : i < \omega\}$ is also homogeneous for L.

Some further notations, notably $a(M)$, $l(M)$, $l(L)$, and (M, A) for $A \subseteq M^*$ will be explained in the next section. The *rank* $r(M)$ of a structure will be defined in §4. The notion of *quotient structure* is defined at the end of the next section as is the notion of *transitiveness* of structures.

2. Permutational structures

A *permutational structure* is a pair $\langle M, G\rangle$ where $M \neq \emptyset$ and $G \subseteq \mathrm{perm}(M)$, where by this notation we mean that G is a subgroup of perm(M), the symmetric group on the universe M. Two permutational structures $\langle M_i, G_i\rangle$ $(i < 2)$ are isomorphic if there is a bijection $\alpha : M_0 \to M_1$ such that $\alpha^{-1}\pi\alpha \in G_0$ if and only if $\pi \in G_1$ $(\pi \in \mathrm{perm}(M_1))$. Just as for relational structures we shall use the same symbol M to denote a permutational structure and its universe.

There is a natural map $M \mapsto \langle M, \mathrm{aut}(M)\rangle$ of the class of relational structures onto the class of permutational structures. Even when we restrict consideration to $\boldsymbol{H}(L)$ for L a finite relational language this natural mapping is not one-one, e.g. the graphs $\bar{K}_m[K_n]$, $K_m[\bar{K}_n]$ are distinct relational structures but yield the same permutational structure. However, there exists $n_L < \omega$ such that at most n_L

structures in $\boldsymbol{H}(L)$ give rise to the same permutational structure. For example, when L has just one binary relation symbol apart from equality we can take $n_L = 2^5 \cdot 4!$.

If M is a relational structure, then by $L(M)$ we mean the given language of the structure. If M is a permutational structure, then by $L(M)$ we mean the language obtained as follows. For each $n < \omega$ and each orbit P of n-tuples take an n-ary relation symbol R_P whose interpretation on M is to be P. Let $L_i(M)$ be the language consisting of all the R_P of finite arity $\leqq i$, where $i \leqq \omega$. Let $a(M)$ the *arity of* M be the least $j \leqq \omega$ such that for all $i < \omega$ and $R \in L_i(M)$ there is a quantifier-free formula ϕ of $L_j(M)$ such that

$$M \models \forall x_0 \cdots \forall x_{i-1}[R(x_0, \ldots, x_{i-1}) \leftrightarrow \phi].$$

Let $L(M)$ denote $L_{a(M)}(M)$. We call $L(M)$ the *canonical language* of the permutational structure M.

The *arity* of a relational structure M is defined to be the arity of the corresponding permutational structure and is also denoted $a(M)$. The *canonical language* of a relational structure M denoted $L^c(M)$ is defined to be the canonical language of the corresponding permutational structure. Note that, if M is a relational structure, then $L^c(M)$ is definable in $L(M)$ iff M is atomic, i.e. for $n < \omega$ every n-type p over $\varnothing$ realized in M is isolated.

Let $\boldsymbol{P}$ denote the class of all relational structures M such that each of $L(M)$ and $L^c(M)$ is definable in the other. Clearly $\boldsymbol{C} \subseteq \boldsymbol{P}$. Let S denote the successor relation on the integers Z and also on ω. The structure $\langle Z, S\rangle \in \boldsymbol{P} - \boldsymbol{C}$ and $\langle \omega, S\rangle$ is atomic but not in $\boldsymbol{P}$. Let $\boldsymbol{P}(L)$ denote the intersection of $\boldsymbol{P}$ with the class of L-structures.

In this paper we are concerned almost exclusively with structures in $\boldsymbol{P}$ and so there is almost no need to distinguish between relational structures and the corresponding permutational ones. Thus where is it convenient we shall forget the distinction. This has two advantages. It eliminates trivial differences such as that between $\bar{K}_m[K_n]$ and $K_m[\bar{K}_n]$, and it makes it easier to define certain structures. For example, suppose $M \in \boldsymbol{C}$, $n < \omega$, E_0 and $E_1 \in \mathscr{E}(M^n)$, $\mathrm{fld}(E_0) = \mathrm{fld}(E_1)$, $E_1 \subseteq E_0$, and $C \in M^n/E_0$. We wish to regard C/E_1 as a structure. We let C/E_1 also denote the permutational structure $\langle C/E_1, G\rangle$, where $G \subseteq \mathrm{perm}(C/E_1)$ is the group of all permutations of C/E_1 which are induced by automorphisms of M. We can endow C/E_1 with its canonical language thus obtaining a relational structure, also denoted C/E_1, in $\boldsymbol{C}$. Similarly, if $M \in \boldsymbol{C}$ and $A \subseteq M^*$ is finite, the structure (M, A) is defined to be $\langle M, G\rangle$ where

$$G = \{\alpha \in \mathrm{aut}(M) : A \subseteq \mathrm{fix}(\alpha)\}.$$

Recall that H consists of all stable structures $M \in C$ with $a(M) < \omega$. For $M \in \boldsymbol{H}$ let $l(M)$ denote the Gödel number of $\langle a(M), n_1, \ldots, n_{a(M)} \rangle$, where $n_i = |s_i(\emptyset)|$ is the number of orbits of i-tuples. Thus $l(M)$ is the Gödel number of the canonical language of M. If L is a finite relational language let $l(L)$ be a number computed effectively from L such that $l(L) > l(M)$ for all $M \in \boldsymbol{H}(L)$.

In keeping with the terminology of permutation groups we say that M is *transitive* if for any $a, b \in M$ there exists $\alpha \in \mathrm{aut}(M)$ such that $\alpha(a) = b$. In writing "aut(M)" we are thinking of M as a relational structure. If M is given as $\langle M, G \rangle$ with $G \subseteq \mathrm{perm}(M)$, then we write "aut($M$)" meaning G.

Let $M \in C$. A *quotient structure* of M is any one of the form C/E_1 where E_0, $E_1 \in \mathscr{E}(M)$, $E_1 \subseteq E_0$, and $C \in M/E_0$. We have specified the structure to be that conferred by the group of permutations of C/E_1 induced by automorphisms of M. Of course, there is no difficulty in defining the notion of quotient structure of M for arbitrary structures M. One can take as given relations all those relations on C/E_1 which are $\{C\}$-definable in M^*.

3. Prime models

At certain points in the paper we shall have a use for the notion of prime model. By convention all structures are countable and so there is no difference between prime models and atomic ones. We now list some basic results. The first two are due to Vaught [13] and the third to Morley [10].

LEMMA 3.1. *If M is prime and $A \subseteq M$ is finite, then M is prime over A.*

LEMMA 3.2. *If $A \subseteq M_i$ and M_i is prime over A for $i < 2$, then there is an isomorphism $\alpha : M_0 \to M_1$ with $A \subseteq \mathrm{fix}(\alpha)$.*

LEMMA 3.3. *If M is prime and $\aleph_0$-stable and $A \subseteq M$ is B-definable for some finite $B \subseteq A$, then M is prime over A.*

An easy consequence of 3.2 and 3.3 is

LEMMA 3.4. *If M is prime and $\aleph_0$-stable, $A \subseteq M$ is B-definable for some finite $B \subseteq A$, and $\pi \in \mathrm{perm}(A)$ is an elementary map, then π can be extended to $\alpha \in \mathrm{aut}(M)$.*

In particular, any automorphism of a 0-definable full substructure of M can be extended to an automorphism. The next lemma shows that when A is indiscernible definability is enough.

LEMMA 3.5. *Let M be prime and $\aleph_0$-stable and $A \subseteq M$ be a definable indiscernible set. Any $\pi \in \mathrm{perm}(A)$ can be extended to $\alpha \in \mathrm{aut}(M)$.*

PROOF. The case in which A is finite is trivial. Thus suppose A is infinite. The idea of the proof is as follows. We shall find finite $B \subseteq A$ and B-definable $A' \supseteq A - B$ such that A' is indiscernible in $M' =_{\mathrm{dfn}} (M, B)$. Applying 3.4 to M' and A', each $\alpha \in \mathrm{perm}(A)$ with $B \subseteq \mathrm{fix}(\alpha)$ can be extended to $\beta \in \mathrm{aut}(M') \subseteq \mathrm{aut}(M)$. Since A is indiscernible, any subset of A of cardinality $|B|$ will do for B. Each $\gamma \in \mathrm{perm}(A)$ can be written as $\alpha_0\alpha_1$ where $|\mathrm{fix}(\alpha_i)| \geqq |B|$. Since α_0, α_1 both extend to automorphisms of M so does γ.

Now we see how to find B and A'. Suppose that in some $N \succcurlyeq M$ there exist $\bar{d}$ and a formula $\psi(x;\bar{u})$ such that $A \cap \psi(M;\bar{d})$ and $A - \psi(M;\bar{d})$ are both infinite. Then $|s_{l(\bar{d})}(A)| = 2^{\aleph_0}$, which contradicts the stability of M. Therefore A is strongly minimal. By normalization [6, §4] there are a formula $\phi(x;\bar{y})$ and $\bar{b} \in M$ such that the symmetric difference of A and $\phi(M;\bar{b})$ is finite and for any $\bar{b}' \in M$ either $\phi(M;\bar{b}') = \phi(M;\bar{b})$ or $A \cap \phi(M;\bar{b})$ is finite. By the Compactness Theorem there exists $n < \omega$ such that $|A \cap \phi(M;\bar{b}')| < n$ whenever $A \cap \phi(M;\bar{b}')$ is finite. We conclude that ϕ and $\bar{b}$ can be chosen so that $\bar{b} \in A$. Let $B = \mathrm{rng}(\bar{b})$.

Since every $a \in A - B$ realizes the same type p over B, $\phi(M;\bar{b}) \supseteq A - B$. Let $A' = \{a \in M : \mathrm{tp}(a \mid \bar{b}) = p\}$ and $M' = (M, B)$. Let $|A' - A|$ be denoted m, which is finite since $A - B \subseteq A' \subseteq \phi(M;\bar{b})$. Clearly A' is a 0-definable strongly minimal subset of M' and $A - B$ is an independent subset of A' since it is infinite and indiscernible. (An infinite subset of a 0-definable strongly minimal set is independent iff it is indiscernible.) Towards a contradiction suppose A' is not independent in M'. There exists finite $C \subseteq A'$ such that $A' \cap \mathrm{acl}_M(C) \not\subseteq C$ and C is independent. Choose $C_i \subseteq A - B$ $(0 \leqq i \leqq m)$ pairwise disjoint such that $|C_i| = |C|$, then the sets $A' \cap \mathrm{acl}_{M'}(C_i)$ are pairwise disjoint by the independence of $A - B$. Hence for some i, $0 \leqq i \leqq m$,

$$((A' \cap \mathrm{acl}_{M'}(C_i)) - C_i) \cap A \neq \varnothing.$$

This contradicts the independence of A. Therefore A' is indiscernible and the proof is complete.

4. Rank

Throughout this section M is an arbitrary member of $\boldsymbol{H}$. Let $k < \omega$, $A \subseteq M$, and $p \in s_k(A)$. We define $r(p) \in \omega$ by defining when $r(p) \geqq n$ simultaneously for all A and p by induction on n: (i) $r(p) \geqq 0$, (ii) $r(p) \geqq n+1$ if there exist B and distinct $p_0, p_1 \in s_k(B)$ such that $A \subseteq B \subseteq M$, $r(p_i) \geqq n$, and $p_i \supseteq p$ for $i < 2$. Then $r(p)$ is the *complete rank* $\mathrm{CR}(p, 2)$ of Shelah [12, p. 55]. Define $r(M^k) =$

$\sup\{r(p) : p \in s_k(\varnothing)\}$. We call $r(M)$ the *rank* of the structure M. This is the main notion of rank used in the paper.

On several occasions we use the Morley rank of a structure by which we mean the Morley rank of its universe. Notice that the Morley rank of M is always $\leqq r(M)$.

Another kind of rank is defined as follows. Let $A \subseteq M^*$ and Δ be a finite set of $*$-formulas. (Recall that a $*$-formula is one of the form $\phi(x; \bar{y})$ where x runs through M^* and the y_i's run through M.) We define $R(A, \Delta, 2)$ by defining inductively when $R(A, \Delta, 2) \geqq n$. Firstly, $R(A, \Delta, 2) \geqq 0$ if $A \neq \varnothing$. Secondly, $R(A, \Delta, 2) \geqq n + 1$ if there exists $B \subseteq M^*$ defined by an instance of a formula in Δ such that $R(A \cap B, \Delta, 2)$ and $R(A - B, \Delta, 2)$ are both $\geqq n$. This is essentially the definition of [12, p. 21]. We shall also use the analogous notion $R(A, \Delta, 2)$, where $A \subseteq M^k$ and Δ is a finite set of M^k-formulas.

In [12, II, 2.2] it is shown that for a stable structure M we cannot have $R(A, \Delta, 2) \geqq n$ for all $n < \omega$. Therefore $R(A, \Delta, 2)$ is always a natural number.

Recall that an M^k-formula is one of the form $\phi(\bar{x}; \bar{y})$ where $\bar{x} = \langle x_0, \dots, x_{k-1} \rangle$ and that a set $\Delta = \{\phi_i(\bar{x}; \bar{y}_i) : i < m\}$ of M^k-formulas is a *basis* if every M^k-formula $\phi(\bar{x}; \bar{y})$ can be expressed as a Boolean combination of formulas $\psi(\bar{x}; \bar{y})$ each of which is obtained from some $\phi_i(\bar{x}; \bar{y}_i)$ by substituting a member of $\bar{y}$ for each member of $\bar{y}_i$.

The following is straightforward so we omit the proof.

LEMMA 4.1. *Let $k < \omega$ and Δ be a finite basis for the M^k-formulas, then* $r(M^k) \leqq R(M^k, \Delta, 2)$.

This shows that $r(M^k)$ is a natural number, i.e. we cannot have $r(M^k) \geqq n$ for all $n < \omega$. There certainly is a finite basis for the M^k-formulas because we are supposing that $M \in \boldsymbol{H}$.

This section is devoted to proving that $r(M^k)$ can be bounded in terms of $l(M)$, $r(M)$, and k, and that $R(A, \Delta, 2)$ can be bounded in terms of $l(M)$, $r(M)$, $[A]$, and $[\Delta] = \sup\{l(\bar{y}) : \phi(\bar{x}; \bar{y}) \in \Delta\}$.

We first prove some facts about finite binary trees which we call trees for short. By ${}^{\leq n}2$ we mean the set of all strings of 0's and 1's of length $\leqq n$. There is a unique empty string. For strings η, ζ we write $\eta \subseteq \zeta$ if ζ extends η. A *tree* $T = \langle T, < \rangle$ is a finite partial ordering isomorphic to $\langle {}^{\leq n}2, \subseteq \rangle$ for some $n < \omega$; n is called the ***height*** of the tree and is denoted $h(T)$.

A subset of $A \subseteq T$ is *closed* if

$$(x \in T,\ y \in A, \text{ and } x \preccurlyeq y) \rightarrow x \in A.$$

A tree T is a *tree of types* if, for some $k<\omega$, T is a set of consistent k-types over subsets of M, $\preccurlyeq$ is $\subseteq$, and if $p,q\in T$ are $\preccurlyeq$-incomparable then $p\cup q$ is inconsistent. If p is a k-type and T is a tree of k-types, we say T is *consistent with* p if $p\cup q$ is consistent for all q in T.

In this section we are dropping our convention that types are complete except when the content requires otherwise. Thus the types just referred to may be incomplete.

LEMMA 4.2. *There exists $F:\omega^2\to\omega$ such that, if $A_0,\dots,A_{m-1}$ are closed subsets of the universe of a tree T with $\bigcup\{A_i : i<m\}=T$ and $h(T)\geqq F(m,n)$, then there exists $T'\subseteq T$ such that $h(T')=n$ and $T'\subseteq A_i$ for some $i<m$.*

PROOF. It is sufficient to treat the case $m=2$ because for $m>2$ the result follows by induction. Let $h(T)=k$. Without loss of generality $|A_0|\geqq 2^k$. We shall show that if k is large enough then there is a subtree $T'\subseteq T$ with $h(T')=n$ and $T'\subseteq A_0$. For $t\in T$ let

$$i(t)=|\{a\in A_0 : t\preccurlyeq a\}|.$$

Let $t_{\langle\ \rangle}$ be the root of T. Suppose $\eta\in{}^{<n}2$ and that t_η has been found in A_0. If possible choose $\preccurlyeq$-incomparable $t_{\eta^\frown\langle 0\rangle}, t_{\eta^\frown\langle 1\rangle}\succcurlyeq t_\eta$ in A_0 so as to maximize

$$j(\eta)=\min\{i(t_{\eta^\frown\langle 0\rangle}),\ i(t_{\eta^\frown\langle 1\rangle})\}.$$

Note that $t_{\eta^\frown\langle 0\rangle}$, $t_{\eta^\frown\langle 1\rangle}$ certainly can be chosen if $i(t_\eta)>k+1$. We can find $t_0=t_\eta, t_1, t'_1,\dots,t_{l-1},t'_{l-1}\in T$ such that t_{l-1}, t'_{l-1} are $\preccurlyeq$-maximal, and $i(t_{j+1})\geqq i(t'_{j+1})$ and t_{j+1}, t'_{j+1} are the immediate successors of t_j for all $j<l-1$. For every $t\succcurlyeq t_\eta$ there exists $i<l$ such that either $t=t_i$ or $i>0$ and $t\succcurlyeq t'_i$. Hence

$$i(t_\eta)\leqq(l-1)\cdot j(\eta)+l\leqq k\cdot j(\eta)+(k+1).$$

Thus $j(\eta)\geqq G_k(i(t_\eta))$ where

$$G_k(u)=\frac{u-(k+1)}{k}\ .$$

Let $G_k^{(j)}$ be the j-th iterate of G_k. Given n choose k large enough such that $G_k^{(n-1)}(2^k)>k+1$. By induction on m for $m<n$, if $l(\eta)=m$, then $i(t_\eta)\geqq G_k^{(m)}(2^k)>k+1$ and $t_{\eta^\frown\langle 0\rangle}$, $t_{\eta^\frown\langle 1\rangle}$ are defined. Letting T' be the subtree of T with universe $\{t_\eta : \eta\in{}^{\leqq n}2\}$ we are done.

LEMMA 4.3. *There is a function $F:\omega^4\to\omega$ such that, if T is a tree of k-types over subsets of M, $A\subseteq M$, $|A|<\omega$, and $h(T)>F(k,l(M),n,|A|)$, then there exist $p\in s_k(A)$ and $T'\subseteq T$ such that $h(T')\geqq n$ and T' is consistent with p.*

PROOF. Let $s_k(A) = \{p_i : i < m\}$ and $A_i = \{q \in T : p_i \cup q$ is consistent$\}$. Then the A_i are closed and cover T. Also, m can be bounded in terms of k, $l(M)$, and $|A|$. The conclusion now follows immediately from 4.2.

LEMMA 4.4. *There exists $G : \omega^4 \to \omega$ such that, if $m > 0$ and T is a tree of k-types over subsets of M, such that for any incomparable $p, q \in T$ there exists $B \subseteq M$ with $|B| \leqq m$ and $(p \cup q) \restriction B$ inconsistent, and $h(T) > G(k, l(M), m, n)$, then there exist $A_i \subseteq M$ $(i \leqq n)$ and types $p_\eta \in s_k(A_{l(\eta)})$ $(\eta \in {}^{\leqq n}2)$ such that $A_i \subseteq A_{i+1}$ $(i < n)$ and $p_\eta \subseteq p_\zeta$ iff $\eta \subseteq \zeta$ $(\eta, \zeta \in {}^{\leqq n}2)$.*

PROOF. We describe how to choose A_i and p_η, and then verify that the method works whenever $h(T)$ is sufficiently large. Let $A_0 = \emptyset$. Choose $\langle p_{\langle\ \rangle}, T_{\langle\ \rangle}\rangle$ such that $p_{\langle\ \rangle} \in s_k(\emptyset)$ and $T_{\langle\ \rangle}$ is a subtree of T consistent with $p_{\langle\ \rangle}$ so as to maximize $h(T_{\langle\ \rangle}) = h_0$. Let $i < n$ and suppose that $A_i \subseteq M$, $p_\eta \in s_k(A_i)$, and a subtree $T_\eta \subseteq T$ have been chosen for all $\eta \in {}^i2$ such that $|A_i| < m \cdot 2^i$, T_η is consistent with p_η and $h(T_\eta) = h_i$. For $\eta \in {}^i2$ let q^0_η, q^1_η be the two immediate successors of the root of T_η, and $B_\eta \subseteq M$ be chosen according to the hypothesis such that $|B_\eta| \leqq m$ and $(q^0_\eta \cup q^1_\eta) | B_\eta$ is inconsistent. Let $A_{i+1} = A_i \cup \bigcup\{B_\eta : \eta \in {}^i2\}$ then certainly $|A_{i+1}| < m \cdot 2^{i+1}$. Now choose $\langle p_\eta, T_\eta\rangle$ simultaneously for all $\eta \in {}^{(i+1)}2$ such that for all $\eta \in {}^i2$ and $j < 2$, $p_{\eta^\frown\langle j\rangle} \in s_k(A_{i+1})$, $p_{\eta^\frown\langle j\rangle} \supseteq p_\eta \cup q^j_\eta$, $T_{\eta^\frown\langle j\rangle}$ is a subtree of T_η consistent with $p_{\eta^\frown\langle j\rangle}$ and $h(T_{\eta^\frown\langle j\rangle}) = h_{i+1}$. The choice is made so as to maximize h_{i+1} the common height of the trees $T_{\eta^\frown\langle j\rangle}$. The procedure is clearly successful unless $h_i = 0$ for some $i < n$. Let T'_η be the tree of k-types with universe $\{p \cup p_\eta : p \in T_\eta,\ q^j_\eta \subseteq p\}$ then $h(T'_\eta) + 1 = h(T_\eta) = h_{l(\eta)}$. Let $F = \omega^4 \to \omega$ be the function of 4.3 which may be supposed increasing in each argument. Applying 4.3 to T we have

$$h(T) \leqq F(k, l(M), h_0 + 1, 0)$$

by maximality of h_0, and applying 4.3 simultaneously to all T^j_η $(\eta \in {}^i2, j < 2)$ for $i \leqq n$ we have

$$h_i - 1 < F(k, l(M), h_{i+1} + 1,\ m \cdot 2^{i+1})$$

by maximality of h_{i+1}. Thus if $h_i = 0$ for some $i < n$ then $h(T)$ can be bounded in terms of k, $l(M)$, m, and n. This completes the proof.

We now come to one of the main results of this section.

LEMMA 4.5. *There exists $F : \omega^3 \to \omega$ such that $r(M^k) \leqq F(l(M), r(M), k)$ for all $k < \omega$.*

PROOF. It suffices to show that for fixed k, $l(M)$ if $r(M^k)$ is very large so is

$r(M)$. We proceed by induction on k. Suppose $r(M^{k+1})$ is very large, then there is a tree T of $(k+1)$-types over subsets of M such that $h(T)$ is very large and for any incomparable $p, q \in T$ there exists $A \subseteq M$ with $|A| < a(M)$ and $(p \cup q) \mid A$ inconsistent. From 4.4, since $h(T)$ is very large there exist very large n, $A_i \subseteq M$ $(i \leqq n^2)$ and $p_\eta \in s_{k+1}(A_{l(\eta)})$ $(\eta \in {}^{\leqq n^2}2)$ such that $p_\eta \subseteq p_\zeta$ iff $\eta \subseteq \zeta$ $(\eta, \zeta \in {}^{\leqq n^2}2)$. Let p'_η be the 1-type of the first component of any $(k+1)$-tuple realizing p_η. There are two cases.

Case 1. There exist $\eta \in {}^{\leqq (n^2-n)}2$ and q such that $p'_\zeta = q$ for all $\zeta \in {}^{(l(\eta)+n)}2$, $\zeta \supseteq \eta$. Let a realize q. For each $\zeta \in {}^{\leqq n}2$ choose $b_1, \dots, b_k$ such that $\langle a, b_1, \dots, b_k \rangle$ realizes $p_{\eta^\frown\zeta}$ and let

$$q_\zeta = \mathrm{tp}(b_1, \dots, b_k\,; A_{l(\eta^\frown\zeta)} \cup \{a\}).$$

Now $\{q_\zeta : \zeta \in {}^{\leqq n}2\}$ is the universe of a tree of types witnessing that $r(M^k) \geqq n$.

Case 2. Otherwise. Then there is an embedding $\sigma : \{{}^{\leqq n}2, \subseteq\} \to \{{}^{\leqq n^2}2, \subseteq\}$ such that $l(\sigma(\zeta)) = n \cdot l(\zeta)$ and $\zeta \mapsto p'_{\sigma(\zeta)}$ is one-to-one. The tree of types with universe $\{p'_{\sigma(\zeta)} : \zeta \in {}^{\leqq n}2\}$ witnesses that $r(M) \geqq n$, so we are done.

We next show that Δ-rank can be bounded in terms of $r(M)$.

Lemma 4.6. *There is a function $F : \omega^4 \to \omega$ such that if Δ is a finite set of *-formulas and $A \subseteq M^*$ then*

$$R(A, \Delta, 2) \leqq F(l(M), r(M), [A], [\Delta]).$$

Proof. Let $[A] = k$. By definition of $[A]$ there exists $E \in \mathscr{E}(M^k)$ such that there is a 0-definable bijection between A and some subset of M^k/E. In the present context this allows us to assume that $A \subseteq M^k$ and that Δ is a finite set of M^k-formulas. Let $x = \langle x_0, \dots, x_{k-1} \rangle$. If $R(A, \Delta, 2) = j$ we can find a tree T of k-types such that $h(T) = j$ and for any incomparable $p, q \in T$ there is a Δ-instance $\phi(\bar{x}\,; \bar{b})$ such that one of $\phi(\bar{x}\,; \bar{b})$, $\neg\,\phi(\bar{x}\,; \bar{b})$ is in p and the other in q. Note that $l(\bar{b}) \leqq [\Delta]$. Let $G : \omega^4 \to \omega$ be the function of 4.4. If $j > G(k, l(M), [\Delta], n)$ we have the conclusion of 4.4 which implies $r(M^k) \geqq n$. Since $r(M^k)$ can be bounded in terms of k, $l(M)$, and $r(M)$ by 4.5, $R(A, \Delta, 2)$ can be bounded in terms of k, $l(M)$, $r(M)$, and $[\Delta]$ which is what had to be proved.

5. Naming elements of M^*

In this section we show that if $M \in \boldsymbol{H}$ and $b \in M^*$ then $M' = (M, \{b\}) \in \boldsymbol{H}$ and moreover $l(M')$ and $r(M')$ can be bounded in terms of $l(M)$, $r(M)$, and $[b]$.

LEMMA 5.1. *There is a function* $F:\omega^3\to\omega$ *such that for every* $b\in M^*$, $a(M,\{b\})\leqq F(l(M),r(M),[b])$.

PROOF. Since $l(M^m)$ and $r(M^m)$ can both be bounded in terms of $l(M)$ and $r(M)$ it is sufficient to consider the case $[b]=1$. Let $b\in M/E$ where $E\in\mathscr{E}(M)$ and Δ be a basis for the M-formulas such that $[\Delta]<a(M)$. Let $R(b,\Delta,2)=k$ where b is seen as a subset of M. By 4.6 k can be bounded in terms of $l(M)$ and $r(M)$. Let $B\subseteq M$ and $\psi(x;\bar b)$ be a conjunction of B-instances of formulas in Δ and of negations of such B-instances such that $b\cap\psi(M;\bar b)\neq\emptyset$ and for every B-instance $\phi(x;\bar b')$ of a formula in Δ

$$\phi(M;\bar b')\cap b\cap\psi(M;\bar b)=\emptyset\quad\text{or}\quad\phi(M;\bar b')\supseteq b\cap\psi(M;\bar b).$$

From the definition of Δ-rank $\psi(x;\bar b)$ can be found with $\leqq k$ conjuncts. Hence $l(\bar b)\leqq k\cdot[\Delta]$. Notice that every $c\in b\cap\psi(M;\bar b)$ realizes the same type, p say, in $s_1(B)$. Let $(M,\{b\})$ be denoted M' and Γ be the set of all M'-formulas of the forms

$$\phi(M;\bar y')\cap b\cap\theta(M;\bar y)=\emptyset,\qquad\phi(M;\bar y')\supseteq b\cap\theta(M;\bar y)$$

where $\phi(x;\bar y')\in\Delta$ and $\theta(x;\bar y)$ is an M-formula with $l(\bar y)\leqq k\cdot[\Delta]$. Let Θ be the set of all true sentences which are B-instances of formulas in $\Gamma\cup\Delta$. In particular, for each B-instance $\phi(x;\bar b')$ of a formula in Δ, one of the formulas

$$\phi(M;\bar b')\cap b\cap\psi(M;\bar b)=\emptyset,\qquad\phi(M;\bar b')\supseteq b\cap\psi(M;\bar b)$$

belongs to Θ. For any $c\in b\cap\psi(M;\bar b)$, Θ implies one of $\phi(c;\bar b')$ and $\neg\phi(c;\bar b')$, which one does not depend on c. Thus Θ implies that there exists $c\in b$ such that $\mathrm{tp}(c;B)=p$, i.e. Γ fixes $\mathrm{tp}(B)$ in M'. But $B\subseteq M$ was arbitrary whence $a(M')\leqq(k+1)\cdot[\Delta]$ can be bounded in terms of $l(M)$, $r(M)$.

LEMMA 5.2. *There is a function* $F:\omega^3\to\omega$ *such that if* $b\in M^*$ *and* $M'=(M,\{b\})$ *then* $l(M')$, $r(M')\leqq F(l(M),r(M),[b])$.

PROOF. From the previous lemma $a(M')$ can be bounded in terms of $l(M)$, $r(M)$, $[b]$. Further, for each i the number of i-types over $\emptyset$ in M' can be bounded in terms of $l(M)$ and $[b]$. Therefore $l(M')$ can be bounded in terms of $l(M)$, $r(M)$, and $[b]$. Let Δ' be a basis for the M'-formulas with $[\Delta']$ bounded in terms of $l(M)$, $r(M)$, and $[b]$. Let $[b]=n$ and $\bar b\in M^n$ represent b. Form Δ from Δ' by replacing each formula $\phi'(x;\bar y)\in\Delta'$ by an M-formula $\phi(x;\bar y,\bar z)$ such that $l(\bar z)=n$ and $\phi'(x;\bar y)$ is equivalent to $\phi(x;\bar y,\bar b)$ in the sense that the same tuples from M satisfy one formula as satisfy the other. Clearly, $R(M',\Delta',2)\leqq$

$R(M,\Delta,2)$. From 4.1, $r(M')\geqq R(M',\Delta',2)$ and from 4.6, $R(M,\Delta,2)$ can be bounded in terms of $l(M)$, $r(M)$, and $[\Delta]$. Since $[\Delta]=[\Delta']+[b]$ can be bounded in terms of $l(M)$, $r(M)$, and $[b]$ we are done.

6. The number of types over a set

Shelah [12, II, 4.10(4)] shows that if the theory of a structure does not have the independence property then for every $m<\omega$ there exists $n<\omega$ such that for every A with $|A|\geqq 2$ we have $|s_m(A)|\leqq |A|^n$. The version of this result convenient for our purposes is

LEMMA 6.1. *There exists $F:\omega^3\to\omega$ such that if $M\in \boldsymbol{H}$ then*

$$|s_m(A)|\leqq |A|^{F(l(M),r(M),m)}\qquad \textit{for all } A\subseteq M \textit{ with } |A|\geqq 2.$$

PROOF. Let Δ be a basis for the M^m-formulas chosen first so as to minimize $[\Delta]$ and then to minimize $|\Delta|$. Then $[\Delta]$ and $|\Delta|$ can be bounded in terms of m and $l(M)$. Let $A\subseteq M$ and $|A|\geqq 2$. For each $p\in s_m(A)$ choose a maximal sequence $\langle \phi_i(\bar{x};\bar{a}_i): i<k\rangle$ in p such that for $i<k$, $\phi_i(\bar{x};\bar{a}_i)$ is an A-instance or the negation of an A-instance of a formula in Δ, and

$$R\left(\bigcap_{j\leqq i}\phi_j(M^m;\bar{a}_j),\Delta,2\right)<R\left(\bigcap_{j<i}\phi_j(M^m;\bar{a}_j),\Delta,2\right)$$

where $\bigcap_{j<0}\phi_j(M^m;\bar{a}_j)$ is to be interpreted as M^m. Clearly $k\leqq R(M^m,\Delta,2)+1$ and so can be bounded in terms of $l(M)$, $r(M)$, and m by 4.6. Given $\langle\phi_i(\bar{x};\bar{a}_i): i<k\rangle$ we can tell which A-instances of formulas in Δ belong to p, and so we can recover p. But the number of possible sequences is $\leqq k_0\cdot|A|^{k_1}$ where k_0, k_1 can be bounded in terms of $l(M)$, $r(M)$, and m. The conclusion is now immediate.

7. Indiscernible sets and formulas

Indiscernible sets and formulas will play a crucial role in our investigations. If $M\in\boldsymbol{H}$, then $I\subseteq M^*$ is called *indiscernible* if for all tuples $\bar{b}_0$, $\bar{b}_1$ from I without repetitions and with $l(\bar{b}_0)=l(\bar{b}_1)$ we have $\mathrm{tp}(\bar{b}_0)=\mathrm{tp}(\bar{b}_1)$; and I is *strongly indiscernible* if every permutation of I can be extended to an automorphism of M. In our context when $|I|<\omega$ there is no difference between the two notions; in general the second notion is clearly stronger.

Let I, $A\subseteq M^*$, I be indiscernible and A be finite, then $\mathrm{cl}^I(A)$ is the least $B\subseteq I$ if there is one such that $2|B|<|I|$ and $I-B$ is indiscernible over $A\cup B$.

The class of such B's is easily seen to be closed under intersection, whence $\mathrm{cl}^I(A)$ is defined whenever there exists $B \subseteq I$ such that $2|B| < |I|$ and $I - B$ is indiscernible over $A \cup B$. By [12, III, 3.4] if I is infinite and A is finite then $\mathrm{cl}^I(A)$ exists. If A is infinite we let

$$\mathrm{cl}^I(A) = \bigcup\{\mathrm{cl}^I(A_0) : A_0 \subseteq A, |A_0| < \omega\}.$$

If I is 0-definable and indiscernible and A is finite, then $\mathrm{cl}^I(A)$, which is finite and A-definable, is algebraic over A. If moreover I is infinite, then $\mathrm{cl}^I(A)$ is the largest finite A-definable subset of I. Also, if each $\{a\} \subseteq A$ is B-definable, then, obviously, $\mathrm{cl}^I(A) \subseteq \mathrm{cl}^I(B)$.

If $I \subseteq M^*$ and $n < \omega$ we call I *n-indiscernible* if for all tuples $\bar{b}_0$, $\bar{b}_1$ from I with $l(\bar{b}_0) = l(\bar{b}_1) \leqq n$ and without repetitions $\mathrm{tp}(\bar{b}_0) = \mathrm{tp}(\bar{b}_1)$. Observe that if $I \subseteq M^k$ and $n \geqq a(M)$ then I is indiscernible if and only if it is n-indiscernible.

Sets $I_i \subseteq M^*$ $(i < j)$ are called *mutually indiscernible* if for all $\bar{a}_i \in I_i$ $(i < j)$ without repetitions $\mathrm{tp}(\bar{a}_0, \ldots, \bar{a}_{j-1})$ depends only on $\langle l(\bar{a}_i) : i < j\rangle$. *Strong mutual indiscernibility* is defined by analogy with strong indiscernibility.

Two 0-definable indiscernible sets I, $J \subseteq M^*$ are said to be *linked* if there is a 0-definable bijection between them.

We now summarize the principal results to be proved in this section. Let it be understood that $M \in \boldsymbol{H}$, that finite bounds have been fixed for $l(M)$ and $r(M)$, that elements are members of M^* and sets are subsets of M^*.

In 7.2 it is shown that $\mathrm{cl}^I(\{a\})$ exists and $|\mathrm{cl}^I(\{a\})|$ can be bounded in terms of $[a]$, provided $I \subseteq M^k$ for some $k \in \omega$ and $|I|$ is large enough compared to $[a]$ and $[I]$. In 7.4 this is extended to indiscernible sets $I \subseteq M^*$. In 7.5 it is shown that, if $I \subseteq M^*$ is 0-definable and n-indiscernible for n sufficiently large compared with $[I]$, then I is indiscernible. We are indebted to the referee for the proof of 7.3; in the original version of the paper 7.3 and 7.4 were proved only for 0-definable I.

In the latter part of the section we look at the relationship between 0-definable indiscernible sets I, J. The final lemma says that if $I_i \subseteq M^*$ $(i < j)$ are pairwise unlinked 0-definable indiscernible sets of sufficiently large cardinality then I_i $(i < j)$ are mutually indiscernible.

LEMMA 7.1. *There exist $F : \omega^3 \to \omega$ and $G : \omega^5 \to \omega$ such that if $M \in \boldsymbol{H}$, $a \in M^*$, $m < \omega$, $I \subseteq M^*$ is n-indiscernible, and $|I| \geqq n$ where $n = G(l(M), r(M), [I], [a], m)$, then there exists $B \subseteq I$ such that $|B| \leqq F(l(M), r(M), [a])$ and $I - B$ is m-indiscernible over $B \cup \{a\}$. Further, F and G can be chosen so that there is a unique minimal B satisfying the conclusion.*

PROOF. Suppose that I is not m-indiscernible over $\{a\}$ and let $n_1 \leqq m$ be the least number such that I is not n_1-indiscernible over $\{a\}$. Let A_1, b_1, c_1 be such that: $A_1 \subseteq I$, $|A_1| = n_1 - 1$, $b_1, c_1 \in I - A_1$ and $\mathrm{tp}(b_1 | A_1 \cup \{a\}) \neq \mathrm{tp}(c_1 | A_1 \cup \{a\})$. We have $|s_{[a]}(B_1)| \geqq 2$, where $B_1 = A_1 \cup \{b_1, c_1\}$. Indeed, if $\sigma \in \mathrm{aut}(M)$ is such that $A_1 \subseteq \mathrm{fix}(\sigma)$ and $\sigma(b_1) = c_1$, then $\mathrm{tp}(a | A_1 \cup \{c_1\}) \neq \mathrm{tp}(\sigma(a) | A_1 \cup \{c_1\})$.

If $I - B_1$ is m-indiscernible over $B_1 \cup \{a\}$, then we are done. In the other case let $n_2 \leqq m$ be minimal such that $I - B_1$ is not n_2-indiscernible over $B_1 \cup \{a\}$. Let A_2, b_2, c_2 be such that: $A_2 \subseteq I$, $|A_2| = n_2 - 1$, $b_2, c_2 \in I - (B_1 \cup A_2)$ and $\mathrm{tp}(b_2 | B_1 \cup A_2 \cup \{a\}) \neq \mathrm{tp}(c_2 | B_1 \cup A_2 \cup \{a\})$. Then we have $|s_{[a]}(B_2)| \geqq 4$, where $B_2 = A_1 \cup A_2 \cup \{b_1, c_1, b_2, c_2\}$. Indeed, if σ and $\tau \in \mathrm{aut}(M)$ are such that $A_1 \cup A_2 \subseteq \mathrm{fix}(\sigma) \cap \mathrm{fix}(\tau)$, $\sigma(b_1) = c_1$, $\sigma(c_1) = b_1$, $\sigma(b_2) = b_2$, $\sigma(c_2) = c_2$, $\tau(b_1) = b_1$, $\tau(c_1) = c_1$, and $\tau(b_2) = c_2$, then a, $\sigma(a)$, $\tau(a)$, $\sigma\tau(a)$ realize different types over B_2.

Proceeding in this way as long as possible, it is easy to see by induction that we can find subsets B_i of I, $i \in \omega$, such that $|B_i| \leqq (m+1)i$ and such that there are 2^i $[a]$-types over B_i. Thus we get subsets C_i of M such that $|C_i| \leqq (m+1)i[I]$ and $|s_{[a]}(C_i)| \geqq 2^i$. By 6.1 the k-th step of the induction cannot be completed, where k is the least integer satisfying

$$2^k > ((m+1)k[I])^{F(l(M), r(M), [a])}$$

with F given by 6.1. Let the e-th step be the last one which is complete. Now e, and hence $|B_e|$, are bounded by a function of m, $l(M)$, $r(M)$, $[a]$, and $[I]$. Since the $(e+1)$-st step could not be completed, $I - B_e$ is m-indiscernible over $B_e \cup \{a\}$. In order for the desired automorphisms to exist in the course of the induction, I should be $|B_e|$-indiscernible. Taking $B = B_e$, we have the conclusion of 7.1 except that the bound on $|B_e|$ depends on m and $[I]$.

To ensure that there is a unique minimal B satisfying the conclusion (except for the sharper bound on $|B|$) it is enough to choose G so as to force $2|B_e| < |I|$. We can cut I down to J with $B \subseteq J \subseteq I$ such that B is the unique minimal set satisfying the conclusion of the lemma when I is replaced by any set between I and J. Moreover $|J|$ can be bounded in terms of m and $|B|$, and hence in terms of m, $l(M)$, $r(M)$, $[a]$, and $[I]$. By choice of G we ensure the existence of such J which is indiscernible with $|J| \geqq [J] = [I]$. Now a "selects" B from J, whence $|s_{[a]}(J)| \geqq |J|^{|B|}$. Let $A \subseteq M$ be chosen so that every singleton $\subseteq J$ is A-definable and $|A| \leqq [J]|J|$. Then $|s_{[a]}(A)| \geqq |s_{[a]}(J)| \geqq |J|^{|B|}$. Applying 6.1 to $A \subseteq M$ with $m = [a]$ we obtain

$$|J|^{|B|} \leqq ([J]|J|)^{F(l(M), r(M), [a])}$$

where F is from 6.1. This becomes

$$|B|\log|J| \leqq F(l(M), r(M), [a])(\log|J| + \log[J]).$$

Since $|J| \geqq [J]$, $|B|$ is bounded in terms of $l(M)$, $r(M)$, and $[a]$.

COROLLARY 7.2. *There exists $H : \omega^4 \to \omega$ such that if $M \in \boldsymbol{H}$, $a \in M^*$, $k < \omega$, $I \subseteq M^k$ is indiscernible, and $|I| > H(l(M), r(M), k, [a])$, then $\mathrm{cl}^I(\{a\})$ exists and $|\mathrm{cl}^I(\{a\})|$ can be bounded in terms of $l(M)$, $r(M)$, and $[a]$.*

PROOF. Let $m = a((M, \{a\}))$. From 5.1 there exists $F : \omega^3 \to \omega$ such that $m \leqq F(l(M), r(M), [a])$. Let $G : \omega^5 \to \omega$ be the function from 7.1 which we may assume is monotonic increasing in all its arguments. Let

$$H(i_0, i_1, i_2, i_3) = G(i_0, i_1, i_2, i_3, F(i_0, i_1, i_3)).$$

By 7.1, $|I| > H(l(M), r(m), k, [a])$ gives $B \subseteq I$ such that $I - B$ is $a((M, a))$-indiscernible over $B \cup \{a\}$, and $|B|$ is bounded in terms of $l(M)$, $r(M)$, and $[a]$. Clearly $I - B$ is actually indiscernible over $B \cup \{a\}$, so we are done.

The next lemma is rather technical. Its purpose is to allow us to generalize 7.2 to the case in which $I \subseteq M^*$. Recall that a basic subset of M^k is one defined by an atomic or negated atomic formula, and that $S_k(A)$ is the set of all solution sets in M of complete k-types over $A \subseteq M^*$.

LEMMA 7.3. *There is a function $F : \omega^3 \to \omega$ such that, if $M \in \boldsymbol{H}$, $k < \omega$, $E \in \mathscr{E}(M^k)$, $I \subseteq M^k/E$ is $F(l(M), r(M), k)$-indiscernible, and $|I| \geqq F(l(M), r(M), k)$, then there exist $d \in M^*$ and $P \in S_k(\{d\})$ such that $[d] \leqq F(l(M), r(M), k)$, and the following conditions hold:*

(i) $2|(P/E) \cap I| \geqq |I|$,

(ii) *for each basic $C \subseteq M^k$ either $P \cap a \neq \emptyset$ and $C \cap P \cap a = \emptyset$ for at least half the $a \in I$, or $C \supseteq P \cap a \neq \emptyset$ for at least half the $a \in I$,*

(iii) *if $c_0, \ldots, c_{n-1} \in P \cap \bigcup I$ fall in distinct E-classes, then $\mathrm{tp}(c_0, \ldots, c_{n-1} | \{d\})$ depends only on n, and*

(iv) *if I is 0-definable, $B \subseteq M$, $P \in S_k(B \cup \{d\})$, and $c_0, \ldots, c_{n-1}$ are as in* (iii), *then $tp(c_0, \ldots, c_{n-1} | B \cup \{d\})$ depends only on n.*

DISCUSSION. This result can be applied to any sufficiently large and sufficiently indiscernible $I \subseteq M^*$ putting $k = [I]$. To understand the statement of the lemma think of the case $k = 1$ and $|I|$ infinite. Then (i) says that P meets almost every member of I, while (ii) says that, if $A \subseteq M$ is definable, then either $A \cap P \cap a = \emptyset$ for almost all $a \in I$, or $A \supseteq P \cap a$ for almost all $a \in I$. Also, (iii) says that, if $J \subseteq P$ is a set of representatives for the members of I meeting P, then J is indiscernible over $\{d\}$, and (iv) that, if $B \subseteq M$ is such that P is a 1-type over $B \cup \{d\}$, then J just mentioned is indiscernible over $B \cup \{d\}$. For (iv) we need I 0-indiscernible. Using the structure theory developed later, this assumption could be eliminated.

PROOF. We need the following special notation which will not be used later. For finite $D \subseteq M^*$, let $[D]'$ denote the least $i < \omega$ such that $i \geqq |D|$ and there exists $A \subseteq M$ such that each singleton $\subseteq D$ is A-definable and $|A| = i$.

For any finite $D \subseteq M^*$ and any $m < \omega$, define $\mathrm{cl}^m(D)$ to be the smallest $B \subseteq I$ satisfying the conclusion of 7.1 when D plays the role of a, where we assume that I is sufficiently large and sufficiently indiscernible. Then we have:

(A) $I - \mathrm{cl}^m(D)$ is m-indiscernible over $D \cup \mathrm{cl}^m(D)$, $|\mathrm{cl}^m(D)| < |I|/2$, and

$$[\mathrm{cl}^m(D)]' \leqq kF(l(M), r(M), [D]'),$$

where F is given by 7.1.

Let Δ be a basis for the M^k-formulas of least possible cardinality (see §1) such that $[\Delta] = a(M)$, the arity of M. Fix $m = a(M)$. For each finite $D \subseteq M^*$ define

$$A_D = \{P \in S_k(D \cup \mathrm{cl}^m(D)) : P/E \supseteq I - \mathrm{cl}^m(D)\}.$$

Let $D_0 = \varnothing$, $P_0 \in A_\varnothing$ and suppose D_n and $P_n \in A_{D_n}$ have already been chosen. If there exist:

$$D^* \supseteq D_n \quad \text{such that } [D^*] \leqq [D]' + mk,$$

$$P^* \subseteq P_n \quad \text{such that } P^* \in A_{D^*} \text{ and } R(P^*, \Delta, 2) < R(P, \Delta, 2),$$

then we choose such a pair and put $D_{n+1} = D^*$, $P_{n+1} = P^*$. If not, we put $D_{n+1} = D_n$ and $P_{n+1} = P_n$. Clearly the two sequences converge, the number of steps required for convergence being bounded in terms of $R(M^k, \Delta, 2)$, and hence in terms of $r(M)$, $l(M)$, and k. Let D, P be the limits of D_n, P_n. By choice of D and P, we have:

(B) for any finite $D^* \supseteq D$ such that $[D^*]' \leqq [D]' + mk$, there exists unique $P^* \in S_k(D^* \cup \mathrm{cl}^m(D^*))$ such that $P^* \subseteq P$ and $P^* \cap \bigcup(I - \mathrm{cl}^m(D^*)) \neq \varnothing$. Hence there is a unique $P' \in S_k(D^* \cup \mathrm{cl}^m(D))$ such that $P' \subseteq P$ and $P' \cap \bigcup(I - \mathrm{cl}^m(D^*)) \neq \varnothing$.

We will show that, for I sufficiently large and sufficiently indiscernible, $d \in M^*$ coding $D \cup \mathrm{cl}^m(D)$ and P $(\in S_k(\{d\}))$ satisfy the conclusion of 7.3. First, notice that $[d]$ can be bounded by a computable function of $l[M]$, $r(M)$, and k, because we have such a bound for $[D]'$, via the bound on the number of steps required for $\langle (D_n, P_n): n < \omega \rangle$ to converge. Secondly, 7.3 (i) holds by (A) since P_n is chosen such that $P_n/E \supseteq I - \mathrm{cl}^m(D_n)$ at each stage. For 7.3 (ii), let $C = \phi(M; \bar{c}) \subseteq M^k$, where $\bar{c} \in M$ and $\phi(\bar{x}; \bar{z})$ is an M^k-formula with $l(\bar{z}) = l(\bar{c}) = a(M)$. Applying (B) to $D^* = D \cup \mathrm{rng}(\bar{c})$, we find $P^* \in S_k(D^* \cup \mathrm{cl}^m(D^*))$ such that $P^* \supseteq P \cap \bigcup(I - \mathrm{cl}^m(D^*))$. Now $P^* \subseteq \phi(M; \bar{c})$ or $P^* \subseteq \neg\phi(M; \bar{c})$, and $|\mathrm{cl}^m(D^*)| < |I|/2$, so we are done.

For 7.3 (iii) we need:

Claim. *Let* $a_1, \ldots, a_r \in P \cap \bigcup I$ *with* $r \leqq a(M)$. *Suppose* $a_i/E \neq a_j/E$ $(1 \leqq i < j \leqq r)$. *Then*

$$I - (\mathrm{cl}^m(\{d\}) \cup \{a_1/E, \ldots, a_r/E\})$$

is $(m - r)$*-indiscernible over* $\{d\} \cup \{a_1, \ldots, a_r\}$.

Proof of the claim. By induction on r. For $r = 0$ this comes from the definition of $\mathrm{cl}^m(\{d\})$, so we may suppose $r \geqq 1$. If the claim fails for r, we can find: s with $1 \leqq s \leqq m - r$, and

$$b_1, \ldots, b_{s-1}, b_s, b_{s+1} \in I - (\mathrm{cl}^m(\{d\}) \cup \{a_1/E, \ldots, a_r/E\})$$

such that b_s and b_{s+1} realize different types over $\{b_1, \ldots, b_{s-1}, a_1, \ldots, a_{r-1}, a_r, d\}$. Let C denote the set $\{b_1, \ldots, b_{s-1}, a_1, \ldots, a_{r-1}\}$. By the induction hypothesis b_s and b_{s+1} realize the same type over $C \cup \{a_r/E\} \cup \{d\}$. Hence there exists $\sigma \in \mathrm{aut}(M)$ such that $C \cup \{d, a_r/E\} \subseteq \mathrm{fix}(\sigma)$ and $\sigma(b_{s+1}) = b_s$. Now a_r and $\sigma(a_r)$ realize different types, say p_1 and p_2, over $C \cup \{d, b_s\}$. We suppose that d was chosen to code $D \cup \mathrm{cl}^m(D)$ exactly, i.e., such that not only is every singleton $\subseteq D \cup \mathrm{cl}^m(D)$ $\{d\}$-definable, but $\{d\}$ is also $(D \cup \mathrm{cl}^m(D))$-definable. Hence a_r and $\sigma(a_r)$ realise different types over $D^* \cup \mathrm{cl}^m(D)$, where $D^* = D \cup C \cup \{b_s\}$. Let $c \in I - \mathrm{cl}^m(D^*)$. Using the induction hypothesis again, we obtain $\tau \in \mathrm{aut}(M)$ such that $C \cup \{d, b_s\} \subseteq \mathrm{fix}(\tau)$ and $\tau(a_r) = c$. Thus p_1 and p_2 (since $a_r/E = \sigma(a_r)/E$) are both realized in $a \cap P$ for each $a \in I - \mathrm{cl}^m(D^*)$. This contradicts (B) and completes the proof of the claim.

Proof of 7.3 (iii). Towards a contradiction suppose we have $1 \leqq s \leqq a(M)$ and $a_1, \ldots, a_{s+1} \in P \cap (\bigcup I)$ such that $a_i/E \neq a_j/E$ $(1 \leqq i < j \leqq s+1)$ and

$$\mathrm{tp}(a_s \,|\, \{a_1, \ldots, a_{s-1}, d\}) \neq \mathrm{tp}(a_{s+1} \,|\, \{a_1, \ldots, a_{s-1}, d\}).$$

Let $D^* = D \cup \{a_1, \ldots, a_{s-1}\}$. Then a_s and a_{s+1} realise different types over $D^* \cup \mathrm{cl}^m(D)$, say p_1 and p_2. But a_s/E and a_{s+1}/E have the same type over

$D^* \cup \mathrm{cl}^m(D)$ by the Claim. As in the proof of the Claim we see that p_1 and p_2 are realised in $a \cap P$ for each $a \in I - \mathrm{cl}^m(D^*)$. This contradicts (B).

For 7.3 (iv) notice that it is enough to treat the case in which $|B| < a(M)$. Thus suppose $B \subseteq M$, $|B| < a(M)$, and $P \in S_k(B \cup \{d\})$. The idea is to show that we can replace D by $B \cup D$, d by an element of M^* coding $B \cup \{d\}$ exactly, and keep P the same, so that the proof of 7.3 (iii) can be repeated. Note that $[B \cup D \cup \mathrm{cl}^m(B \cup D)]'$ is bounded in terms of $l(M)$, $r(M)$, and k. Let $Q \in S_k(B \cup D \cup \mathrm{cl}^m(B \cup D))$ be chosen so that $Q \cap P \cap a \neq 0$ for at least half the members a of I. Then $Q \cap a \neq \emptyset$ for almost all $a \in I$, and $Q \subseteq P$ since Q is a type over a larger set.

Case 1. Q splits $P \cap a$ for almost all $a \in I$. We can see Q as a Boolean combination of a small number of basic subsets of M^k. One of these basic subsets must split $P \cap a$ for almost all $a \in I$. This contradicts the choice of D and P.

Case 2. Not Case 1, but there exists $a \in I$ such that $P \cap a \neq Q \cap a$. Since Case 1 fails, there are only a few such a. Let the set of all such a be denoted K. Since I is 0-definable, $\mathrm{cl}^m(B \cup D)$ is $(B \cup D)$-definable. It follows that K is $(B \cup D)$-definable, which contradicts $P \in S_k(B \cup D)$.

Since Cases 1, 2 both fail, $Q = P$. Thus D can be replaced by $B \cup D$. Clearly $\mathrm{cl}^m(B \cup D) = \mathrm{cl}^m(D)$, and so d can be replaced by an element coding $B \cup \{d\}$. Repeating the argument of 7.3 (iii) we have the desired conclusion.

COROLLARY 7.4. *There exists $H : \omega^4 \to \omega$ such that if $M \in \mathbf{H}$, $a \in M^*$, $I \subseteq M^*$ is indiscernible, and $|I| \geqq H(l(M), r(m), [I], [a])$ then $\mathrm{cl}^I(\{a\})$ exists and its size can be bounded in terms of $l(M)$, $r(M)$, and $[a]$. If I is 0-definable, $|\mathrm{cl}^I(\{a\})|$ can be bounded in terms of $l(M)$ and $[a]$.*

PROOF. We can suppose I is as in 7.3 with $k = [I]$ and that $|I|$ is large compared with $l(M)$, $r(M)$, k, and $[a]$. In the proof of 7.3 we saw that $I - (P/E) = \mathrm{cl}^m(\{d\})$ has size bounded in terms of $l(M)$, $r(M)$, k, and $[a]$ by 7.1. From 7.3 (iii) there exists $I' \subseteq M^k$, an indiscernible set of representatives for $I \cap (P/E)$. Let $a' = \langle a \rangle ^\frown \bar{b}$ where $\bar{b}$ is a sequence of k-tuples, one representing each element in $I - (P/E)$. By 7.2, $\mathrm{cl}^{I'}(\{a'\})$ exists and its size is bounded in terms of $l(m)$, $r(M)$, and $[a']$. Clearly $((P/E \cap I) - (\mathrm{cl}^{I'}(\{a'\})/E)$ is indiscernible over $\{a'\} \cup (I - (P/E)) \cup (\mathrm{cl}^{I'}(\{a'\})/E)$. Therefore $\mathrm{cl}^I(\{a'\})$ exists. Hence $\mathrm{cl}^I(\{a\})$ exists and its size can be bounded in terms of $l(M)$, $r(M)$, and $[a]$ by the argument used to bound $|B|$ in 7.1.

If I is 0-definable, $\mathrm{cl}^I(\{a\})$ is $\{a\}$-definable. For c conjugate to a in M there are $|\mathrm{cl}^I(\{a\})|+1$ possible values for $|\mathrm{cl}^I(\{a\}) \cap \mathrm{cl}^I(\{c\})|$. Hence there are at least this number of $2[a]$-types over $\varnothing$. But the number of such types can be bounded in terms of $l(M)$ and $[a]$. Therefore $|c^I(\{a\})|$ can be bounded in terms of $l(M)$ and $[a]$ in this case.

COROLLARY 7.4. *There exists* $H : \omega^4 \to \omega$ *such that if* $M \in \boldsymbol{H}$, $a \in M^*$, $I \subseteq M^*$ *is indiscernible and 0-definable and* $|I| \geqq H(l(M), r(M), [I], [a])$, *then* $\mathrm{cl}^I(\{a\})$ *exists. In this case* $|\mathrm{cl}^I(\{a\})|$ *can be bounded in terms of* $l(M)$ *and* $[a]$.

PROOF. We can suppose I is as in 7.3 with $k = [I]$ and that $|I|$ is large compared with $l(M)$, $r(M)$, k, and $[a]$. In the conclusion of 7.3, $[d]$ and hence $|I - (P/E)|$ are small, i.e. can be bounded in terms of $l(M)$, $r(M)$, and k. Also from (iii) there exists $I' \subseteq M^k$ an indiscernible set of representatives for P/E.

Let $a' = \langle a \rangle ^\frown \bar{b}$ where $\bar{b}$ is a sequence of k-tuples one representing each element in $I - (P/E)$. By 7.2, $\mathrm{cl}^{I'}(\{a'\})$ exists and $|\mathrm{cl}^{I'}(\{a'\})|$ can be bounded in terms of $l(M)$, $r(M)$, and $[a']$. Clearly $(P/E) \cap I - (\mathrm{cl}^{I'}(\{a'\})/E)$ is indiscernible over $\{a'\} \cup (I - (P/E)) \cup (\mathrm{cl}^{I'}(\{a'\})/E)$. Therefore $\mathrm{cl}^I(\{a'\})$ exists and hence $\mathrm{cl}^I(\{a\})$ also.

If a and a' are conjugate there are $|\mathrm{cl}^I(\{a\})|+1$ possible values for $|\mathrm{cl}^I(\{a\}) \cap \mathrm{cl}^I(\{a'\})|$. Hence there are at least this number of $2[a]$-types over $\varnothing$.

COROLLARY 7.5. *There exists* $F : \omega^3 \to \omega$ *such that if* $M \in \boldsymbol{H}$ *and* $I \subseteq M^*$ *is 0-definable and* $F(l(M), r(M), [I])$*-indiscernible then* I *is indiscernible.*

PROOF. We can suppose I is as in 7.3 and that $|I|$ is large compared with $l(M)$, $r(M)$, and $[I]$. Arguing as in 7.4 with a empty we have $I' - \mathrm{cl}^{I'}(\{a'\})$ indiscernible over $\{a'\} \cup \mathrm{cl}^{I'}(\{a'\})$. Therefore $(P/E) - (\mathrm{cl}^{I'}(\{a'\})/E)$ is indiscernible over $(I - (P/E)) \cup (\mathrm{cl}^{I'}(\{a'\})/E)$, i.e. there exists $B \subseteq I$ namely $(I - (P/E)) \cup (\mathrm{cl}^{I'}(\{a'\})/E)$ such that $I - B$ is indiscernible over B.

Choose F such that

$$|I - (P/E)| + |\mathrm{cl}^{I'}(\{a'\})| \leqq F(l(M), r(M), [I]).$$

Then $|B| \leqq F(l(M), r(M), [I])$ and by hypothesis any subset of I of cardinality $|B|$ is conjugate to B. The indiscernibility of I is now immediate.

REMARK. 7.5 can be proved without the assumption that I is 0-definable. However, the only proof we know requires the theory from §11. The methods of this section yield only the conclusion: there exists unique minimal $B \subseteq I$, of size bounded in terms of $l(M)$, $r(M)$, and $[I]$, such that $I - B$ is indiscernible over B.

For the rest of the section we shall be studying the relationships between different definable indiscernible sets. Roughly speaking two such sets are either mutually indiscernible or there is a bijection between them which is 0-definable. The first result in this vein is:

LEMMA 7.6. *Let $M \in \boldsymbol{H}$, I, $J \subseteq M^*$ be 0-definable indiscernible sets, and let* $\max\{|I|,|J|\} \geqq 7$. *Then either*

(i) *for all m, n such that $m+1<|I|$ or $n+1<|J|$, and all distinct $a_1,\dots,a_m \in I$ and distinct $b_1,\dots,b_n \in J$, $\mathrm{tp}(a_1,\dots,a_m,b_1,\dots,b_n)$ depends only on m and n; or*

(ii) *there is a unique 0-definable bijection between I and J.*

REMARK. Neither the finiteness of $a(M)$ nor the $\aleph_0$-categoricity of M are needed. It is enough that, if I and J are both infinite, then the formula defining I be stable.

DISCUSSION. Since $M \in \boldsymbol{H}$, indiscernibility and strong indiscernibility are the same for definable subsets by 3.5. Let $\mathrm{alt}(A)$ denote the alternating group on the set A if A is finite, and mean the same as $\mathrm{perm}(A)$ otherwise. By 1_A we denote the identity map on A. A subgroup $G \leqq \mathrm{perm}(A)$ is called *closed* if it satisfies

$$(\forall \alpha \in \mathrm{perm}(A))(\forall \beta \subseteq \alpha)(|\mathrm{dom}(\beta)| < \omega \rightarrow (\exists \gamma \in G)(\beta \subseteq \gamma)) \rightarrow \alpha \in G.$$

When A is finite all subgroups of $\mathrm{perm}(A)$ are closed. For any structure M, $\aleph_0$-categorical or not, $\mathrm{aut}(M)$ is closed. If $A, B \subseteq M$ are 0-definable, then $\mathrm{aut}_M(A,B)$ is a closed subgroup of $\mathrm{perm}(A)$ by 3.4.

Now let I, J satisfy the hypothesis of the lemma. The following easy observations will be helpful.

FACT 1. $\mathrm{aut}_M(I,J)$ is a closed normal subgroup. Hence, either $\mathrm{aut}_M(I,J) = \{1_I\}$ or $\mathrm{aut}_M(I,J) \geqq \mathrm{alt}(I)$ since $|I|>4$.

FACT 2. The following are equivalent:

(i) $\mathrm{aut}_M(I,J) = \mathrm{perm}(I)$,

(ii) I is (strongly) indiscernible over J,

(iii) J is (strongly) indiscernible over I,

(iv) I and J are (strongly) mutually indiscernible.

FACT 3. The following are equivalent:

(i) $\mathrm{aut}_M(I,J) = \{1_I\}$,

(ii) $\{a\}$ is J-definable for each $a \in I$, (*)

(iii) $\mathrm{tp}(a_0 | J) \neq \mathrm{tp}(a_1 | J)$ for all distinct $a_0, a_1 \in I$.

PROOF OF 7.6. From Fact 1 there are two cases. If $\mathrm{aut}_M(I, J) \geqq \mathrm{alt}(I)$, then we have (i) of the conclusion provided $|m+1| < i$. At the same time $\mathrm{aut}_M(I, J) \geqq \mathrm{alt}(I)$ is clearly inconsistent with $\mathrm{aut}_M(J, I) = \{1_J\}$. Hence $\mathrm{aut}_M(J, I) \geqq \mathrm{alt}(J)$, which yields (i) of the conclusion provided $|n+1| < |J|$. Therefore (i) holds if $\mathrm{aut}_M(I, J) \geqq \mathrm{alt}(I)$.

For the rest, suppose $\mathrm{aut}_M(I, J) = \{1_I\}$ and without loss of generality that $|I| \geqq |J|$. The essence of the conclusion in this case is that (*) above can be strengthened to "for all $a \in I$, $\{a\}$ is $\{b\}$-definable for some $b \in J$". Recall from Fact 3 that $\mathrm{tp}(a_0 | J) \neq \mathrm{tp}(a_1 | J)$ if $a_0, a_1 \in I$ are distinct. Fix $a \in I$. It follows that J is not indiscernible over $\{a\}$. Suppose J is infinite. From the superstability of M, $\mathrm{cl}^J(\{a\})$ is finite. Since J is not indiscernible over $\{a\}$, $\mathrm{cl}^J(\{a\}) \neq \emptyset$. Since J is 0-definable, $\mathrm{cl}^J(\{a\})$ is $\{a\}$-definable. If $|\mathrm{cl}^J(\{a\})| > 1$, then there is a non-trivial 2-type on I, because $|\mathrm{cl}^J(\{a_0\}) \cap \mathrm{cl}^J(\{a_1\})|$ takes at least three different values at a_0, a_1 run through I. Therefore $|\mathrm{cl}^J(\{a\})| = 1$ which means there is a 0-definable bijection between I and J as required.

Now suppose J is finite. There is a mapping $\sigma \mapsto \pi$ of $\mathrm{perm}(I)$ into $\mathrm{perm}(J)$ such that $\sigma \cup \pi$ extends to an automorphism of M for all $\sigma \in \mathrm{perm}(I)$. The mapping $\sigma \mapsto \pi$ is one-one, whence $|J| \geqq |I|$. Since $|J| \leqq |I|$ by assumption, we have $|I| = |J|$. From [11, 7.4] the symmetric group on I has no outer automorphism since $|I| > 6$. Hence the bijection between $\mathrm{perm}(I)$ and $\mathrm{perm}(J)$ under discussion must be induced by a bijection between I and J. Since the bijection between $\mathrm{perm}(I)$ and $\mathrm{perm}(J)$ is 0-definable, so is the corresponding bijection between I and J. This completes the proof.

If (ii) holds we say that the definable indiscernible sets I, J are *linked* and if (i) holds we say they are *unlinked.* We shall tacitly assume that all indiscernible sets considered have cardinality $\geqq 7$. Whether I, J are linked or not is invariant under naming $a \in M^*$ provided $|I|$ and $|J|$ are sufficiently large compared with $l(M)$, $r(M)$, $[I]$, $[J]$, and $[a]$. This is now made precise.

LEMMA 7.7. *There is a function $F : \omega^5 \to \omega$ such that if $M \in \boldsymbol{H}$, $I, J \subseteq M^*$ are 0-definable, $a \in M^*$, and*

$$|I|, |J| \geqq F(l(M), r(M), [I], [J], [a])$$

then $I - \mathrm{cl}^I(\{a\})$ and $J - \mathrm{cl}^J(\{a\})$ are linked over $\{a\}$ iff I and J are linked.

PROOF. The "if" part is obvious. Suppose I, J are unlinked but that $I - \mathrm{cl}^I(\{a\})$ and $J - \mathrm{cl}^J(\{a\})$ are linked over a. Let $I_0 \subseteq I$, $J_0 \subseteq I$ have the same power m, where

$$m < |I - \mathrm{cl}^I(\{a\})| = |J - \mathrm{cl}^J(\{a\})|,$$

then $|s_{[a]}(I_0 \cup J_0)| > m!$ because a can define any of the $m!$ possible bijections between I_0 and J_0. Thus if $|I|$ and $|J|$ are large enough there is a contradiction of 6.1.

Let us call sets $I_i \subseteq M^*$ $(i < j)$ *almost mutually indiscernible* if, for all m_i such that $m_i + 1 < |I_i|$ $(i < j)$ and all distinct $a_{i1}, \ldots, a_{im_i} \in I_i$ $(i < j)$,

$$\mathrm{tp}(a_{01}, \ldots, a_{0m_0}, \ldots, a_{j-1\,0}, \ldots, a_{j-1\,m_{j-1}})$$

depends only on $\langle m_i : i < j \rangle$.

LEMMA 7.8. *Let $I_i \subseteq M^*$ $(i < j)$ be 0-definable indiscernible sets which are pairwise unlinked and let $|I_i| \geqq 7$ $(i < j)$, then I_i $(i < j)$ are almost mutually indiscernible.*

PROOF. We can suppose that $|I_i|$ is nondecreasing with i. We employ induction on j. Let $m_0 < |I_0| - 1$ and $a_{01}, \ldots, a_{0m_0} \in I_0$ be distinct. Let $a_{01}, \ldots, a_{0m_0}$ be named, then I_i remains indiscernible for $0 < i < j$. Suppose for a contradiction argument that there exist i, k, $1 \leqq i < k < j$, such that I_i, I_k are linked over $\{a_{01}, \ldots, a_{0m_0}\}$. Let $m \leqq m_0$ be the least number such that I_i, I_k are linked over $\{a_{01}, \ldots, a_{0m}\}$. Then over $\{a_{01}, \ldots, a_{0\,m-1}\}$ there is a unique bijection between I_i and I_k defined by a_{0m}. If I_i, I_k are infinite this contradicts the stability of M. If I_i, I_k are finite there are not enough possibilities for a_{0m} to define all possible bijections between I_i and I_k. Therefore no such i, k exist and so I_i, I_k are unlinked over $a_{01}, \ldots, a_{0m_0}$ for all i, k, $1 < i < k < j$. The lemma follows by applying the induction hypothesis to $(M, \{a_{01}, \ldots, a_{0m_0}\})$.

Starting from the last lemma and using the ideas of 7.3 and 7.5 we can prove:

LEMMA 7.9. *There is a function $F : \omega^3 \to \omega$ such that if $M \in \boldsymbol{H}$ and $I_i \subseteq M^*$ $(i < j)$ are 0-definable indiscernible sets which are pairwise unlinked and*

$$|I_i| \geqq F(l(M), r(M), \max\{[I_i] : i < j\}) \qquad (i < j)$$

then I_i $(i < j)$ are mutually indiscernible.

8. Strongly minimal sets

In this section we show that if $M \in \boldsymbol{H}$ and there is a strongly minimal set $H \subseteq M^*$ then H is disintegrated in the sense of Zilber [14], i.e. after factoring out the coarsest nontrivial equivalence relation which is definable on H we are left with an indiscernible set. We could rely on the theorem of Cherlin and Zilber [3, 2.1] characterizing the dependence relations which can arise from $\aleph_0$-

categorical strongly minimal sets. However, since there is an easy proof from first principles of the theorem we need, we present it.

Call $H \subseteq M^*$ *strongly minimal* if H is 0-definable and infinite, and if for any definable $A \subseteq H$ either A or $H - A$ is finite. We call H *strictly minimal* if H is strongly minimal and there is no nontrivial 0-definable equivalence relation on H. Equivalently $H \subseteq M^*$ is strictly minimal if it is 0-definable, infinite, and realizes a unique nontrivial 2-type. If $M \in \boldsymbol{C}$ and $H \subseteq M^*$ is strongly minimal, then there is a coarsest nontrivial 0-definable equivalence relation E on $H - \operatorname{acl}(\emptyset)$. Clearly $(H - \operatorname{acl}(\emptyset))/E$ is strictly minimal. Thus in an $\aleph_0$-categorical structure associated with each strongly minimal set is a unique strictly minimal set. A strongly minimal set is called *disintegrated* if the associated strictly minimal set is indiscernible. We will assume familiarity with the exchange principle for strongly minimal sets and with the notion of an independent subset (see [1]).

If H is strongly minimal and $A \subseteq H$ then by $(A)_H$ we denote $H \cap \operatorname{acl}(A)$, and by $(A)_H^+$ the set $(A)_H - \bigcup\{(B)_H : B \subsetneqq A\}$.

LEMMA 8.1. *If H is strongly minimal, $A \cup \{a\} \subseteq H$ is independent, $b \in (A)_H^+$, and $c \in (\{a, b\})_H^+$, then $c \in (A \cup \{a\})_H^+$.*

PROOF. Towards a contradiction suppose the hypotheses are true and that the conclusion fails. Then there is $B \subsetneqq A \cup \{a\}$ such that $c \in (B)_H$.

Case 1. $B \subseteq A$. Since $c \in (\{a, b\})_H^+$, by exchange we have $a \in (\{b, c\})_H$ whence $a \in (\{b\} \cup B)_H \subseteq (A)_H$. This contradicts the independence of $A \cup \{a\}$.

Case 2. Otherwise. Then $a \in B$ and there exists $d \in A - B$. Since $c \in (\{a, b\})_H^+$, by exchange we have $b \in (\{a, c\})_H \subseteq (B)_H$. Also $b \in (\{d\} \cup (A - \{d\}))_H^+$, whence by exchange

$$d \in (\{b\} \cup (A - \{d\}))_H \subseteq ((A - \{d\}) \cup \{a\})_H$$

since $B \subseteq (A - \{d\}) \cup \{a\}$. But this contradicts the independence of $A \cup \{a\}$. So we are done.

LEMMA 8.2. *If $M \in \boldsymbol{H}$ and $H \subseteq M^*$ is strictly minimal then H is indiscernible.*

PROOF. Towards a contradiction suppose H is not indiscernible. Choose distinct $a_0, \dots, a_{n-1} \in H$ with n as small as possible such that for some $b_0, b_1 \in H - \{a_0, \dots, a_{n-1}\}$, $\operatorname{tp}(a_0, \dots, a_{n-1}, b_0) \neq \operatorname{tp}(a_0, \dots, a_{n-1}, b_1)$. Since H is strictly minimal $n \geqq 2$. If $n \geqq 2$ then, by naming $a_0, \dots, a_{n-3}$ and considering $H - (\{a_0, \dots, a_{n-3}\})_H$ factored by the coarsest $\{a_0, \dots, a_{n-3}\}$-definable equivalence relation having finite classes, we can reduce to the case $n=2$. Thus suppose $n=2$. This means that, if $A\subseteq H$ and $|A|=2$, then $(A)^+_H\neq\varnothing$.

Let $B\subseteq H$ be an independent set of cardinality $m<\omega$. From 8.1 it follows that for every nonempty $C\subseteq B$ there exists $c\in(C)^+_H$. Also, if $C_0, C_1\subseteq B$ and $C_0\neq C_1$, then $(C_0)^+_H\cap(C_1)^+_H=\varnothing$. If not, let $d\in(C_0)^+_H\cap(C_1)^+_H$. By definition of $(C_i)^+_H$ neither of C_0, C_1 contains the other. Hence there are $c_0\in C_0-C_1$ and $c_1\in C_1-C_0$. By exchange $(C_i)_H=((C_i-\{c_i\})\cup\{d\})_H$ $(i<2)$. Therefore

$$(C_0\cup C_1)_H=((C_0\cup C_1\cup\{d\})-\{c_0,c_1\})_H.$$

But this contradicts the independence of $C_0\cup C_1$ since the set $((C_0\cup C_1)\cup\{d\})-\{c_0,c_1\}$ has one less element. We conclude that over B at least 2^m types are realized in H.

We can suppose that each element of H is represented by a single element of M. Otherwise replace M by a suitable finite cartesian power. We have shown that for every $m<\omega$ there exists $A\subseteq M$ with $|A|=m$ and $|s_1(A)|\geqq 2^m$. This contradicts 6.1 so we are done.

CHAPTER II — SHRINKING AND STRETCHING

9. The coordinatization and trichotomy theorems

From [3] we quote a coordinatization theorem for $\aleph_0$-categorical, $\aleph_0$-stable structures and adapt it to the special context of this paper. An important consequence will be the Trichotomy Theorem 9.4 which says that there is $F:\omega^2\rightarrow\omega$ such that for any $M\in\boldsymbol{H}$ if N is a transitive quotient structure of M then either $|N|\leqq F(l(M),r(M))$, or some member of $\mathscr{E}(N)$ is nontrivial, or N is perfect in the sense defined below. Recall that "transitive" means there is only one 1-type and that N is a quotient structure of M if its universe has the form C/E_1 where for some $E_0,E_1\in\mathscr{E}(M)$, with $E_1\subseteq E_0$, C is an E_0-class and $\mathrm{fld}(E_1)=\mathrm{fld}(E_0)$. The automorphisms of N are all those induced by automorphisms α of M such that $\alpha(C)=C$.

We now say what we mean by a perfect structure. For each triple $\langle F,m,n\rangle$ where F is a finite transitive structure, $3\leqq m\leqq\omega$ and $0<2n<m$, we construct $M(F,m,n)\in\boldsymbol{H}$. First let $M_0(F,m)$ be a structure on which there is a 0-definable equivalence relation E such that $M_0(F,m)/E\simeq F$ and the E-classes are mutually indiscernible sets of cardinality m. Then $M_0(F,m)$ is fixed up to isomorphism. Let $M(F,m,n)$ be the structure whose universe is

$$A = \{X \subseteq M_0(F, m) : |X \cap C| = n \text{ for all } C \in M_0(F, m)/E\}$$

and whose automorphisms are all permutations of A induced by automorphisms of $M_0(F, m)$. A structure is *perfect* if it is isomorphic to one of the structures $M(F, m, n)$.

Notice that if N is perfect its perfectness is *witnessed by* some triple $\langle E_0, E_1, R\rangle$ such that

(i) R is a 0-definable ternary relation on N,

(ii) $E_0, E_1 \in \mathscr{E}(N^2)$, $\mathrm{fld}(E_0) = \mathrm{fld}(E_1)$, and $E_1 \subseteq E_0$,

(iii) $\mathscr{F} = \{C/E_1 : C \in N^2/E_0\}$ is a finite family of mutually indiscernible sets of N^* each having cardinality $m \geqq 3$ and $\mathrm{aut}(N)$ permutes $\mathscr{F}$ transitively,

(iv) R induces a 0-definable relation $S \subseteq N \times \bigcup \mathscr{F}$ such that for some n with $0 < 2n < m$ the mapping $a \mapsto \{b : \langle a, b\rangle \in S\}$ is a bijection from N onto

$$\{X \subseteq \bigcup \mathscr{F} : |X \cap Y| = n \text{ for all } Y \in \mathscr{F}\}.$$

To justify this remark it suffices to define E_0, E_1, R when $N = M(F, m, n)$. Let X, Y denote $\langle X_0, X_1\rangle$, $\langle Y_0, Y_1\rangle$ and define

$$E_0 = \{\langle X, Y\rangle \in A^2 \times A^2 : \exists a, b \in M_0(F, m) \times [\langle a, b\rangle \in E \;\&\; X_0 \cap X_1 = \{a\} \;\&\; Y_0 \cap Y_1 = \{b\}]\},$$

$$E_1 = \{\langle X, Y\rangle \in E_0 : X_0 \cap X_1 = Y_0 \cap Y_1\},$$

$$R = \{\langle Z, X_0, X_1\rangle \in A^3 : |X_0 \cap X_1| = 1 \;\&\; Z \supseteq X_0 \cap X_1\}.$$

We shall not verify that $M(F, \omega, n) \in \boldsymbol{H}$ because we shall not need it and the only proof we know is quite tedious.

In general we do not know whether from the isomorphism type of $M(F, m, n)$ one can recover m, n, and the isomorphism type of F, although this is possible if m is large enough compared with $|F|$ and n. Despite this, in speaking of a perfect structure we shall tacitly assume that some particular perfect representation has been given so that we can talk of its *width*, meaning $|F|$, and its *index*, meaning n. Notice that in $M(F, m, n)$ at least $|F| + 1$ 2-types are realized because for $X_0, X_1 \in A$, $|\{C \in M_0(F, m)/E : X_0 \cap C = X_1 \cap C\}|$ can take any value $\leqq |F|$. Likewise at least $n + 1$ 2-types are realized because for each $i \leqq n$ we can find $X_0, X_1 \in A$ such that $|X_0 \cap X_1 \cap C| = i$ for all $C \in M_0(F, m)/E$.

From [3, Theorem 4.1] we quote without proof the

COORDINATIZATION THEOREM 9.1. *Let $N \in \boldsymbol{C}$ be infinite $\aleph_0$-stable, and transitive. There exists P, the solution set of some type over $\varnothing$ in N^* such that P has Morley rank 1 and $\mathrm{acl}(\{a\}) \cap P \neq \varnothing$ for all $a \in N$.*

REMARK. The proof of 9.1 uses the difficult result of Cherlin and Zilber [3, Theorem 2.1] characterizing all possible dependence relations of $\aleph_0$-categorical strongly minimal sets. In this paper 9.1 is applied only to quotients of structures in $\boldsymbol{H}$ and so all strongly minimal sets are disintegrated. This special case of 9.1 does not rely on the theorem of Cherlin and Zilber just alluded to.

COROLLARY 9.2. *If we add to the hypothesis of* 9.1 *that N is a quotient structure of some $M \in \boldsymbol{H}$, then we can add to the conclusion that there is a finite equivalence relation E on P 0-definable in N such that the E-classes are mutually indiscernible.*

PROOF. Starting with the conclusion of 9.1 we can suppose that on P there is no nontrivial 0-definable equivalence relation with finite classes. For, if there were one, by $\aleph_0$-categoricity there would be a coarsest one and we could factor it out.

By the finite equivalence relation theorem [12, III, 2.8] there exists a finite equivalence relation E on P 0-definable in N such that the equivalence classes have Morley degree 1. Thus the E-classes are strongly minimal and hence strictly minimal because on P no nontrivial equivalence relation with finite classes is 0-definable. By 8.2 the E-classes are indiscernible sets. If two of these indiscernible sets were linked there would be a non-trivial equivalence relation on P with finite classes which was 0-definable. Thus the E-classes are pairwise unlinked. Since the E-classes are infinite they are mutually indiscernible by 7.8.

The Trichotomy Theorem will be proved by applying the Compactness Theorem to the following consequence of 9.1.

COROLLARY 9.3. *If $M \in \boldsymbol{H}$ and N is an infinite transitive quotient structure of M then either*

(i) *some member of $\mathscr{E}(N)$ is nontrivial, or*

(ii) *N is perfect with width and index less than the number of 2-types over $\varnothing$ realized in M.*

PROOF. We start from the conclusion of 9.2. Define

$$R_0 = \{\langle a_0, a_1\rangle \in N^2 : |\mathrm{acl}(\{a_0\}) \cap C| = |\mathrm{acl}(\{a_1\}) \cap C| \text{ for all } C \in P/E\},$$

$$R_1 = \{\langle a_0, a_1\rangle \in N^2 : \mathrm{acl}(\{a_0\}) \cap P = \mathrm{acl}(\{a_1\}) \cap P\}.$$

Obviously $R_0, R_1 \in \mathscr{E}(N)$. If R_0 is universal then, because N is transitive and $\mathrm{aut}(N)$ permutes P/E transitively, $n = |\mathrm{acl}(\{a\}) \cap C|$ is the same for all $a \in N$ and $C \in P/E$. Thus $a \mapsto \mathrm{acl}(\{a\}) \cap P$ $(a \in N)$ is a 0-definable mapping of N into

$$A = \{X \subseteq P : |X \cap C| = n \text{ for all } C \in P/E\}.$$

If R_1 is equality then the mapping is one-one. The mapping is onto because the classes $C \in P/E$ are mutually indiscernible. Thus if R_0, R_1 are both trivial, N is perfect. The width and index are less than the number of 2-types over $\emptyset$ realized in N and hence less than the number realized in M.

TRICHOTOMY THEOREM 9.4. *There exists $F : \omega^2 \to \omega$ such that if $M \in \boldsymbol{H}$ and N is a transitive quotient structure of M then one of the following three possibilities holds*:

(i) $|N| \leqq F(l(M), r(M))$,

(ii) *there is a nontrivial member of $\mathscr{E}(N)$*,

(iii) *N is a perfect structure.*

PROOF. Fix $r < \omega$, a finite relational language L, a unary relation symbol U of L, and a binary relation symbol R of L. Let τ be an infinite conjunction of L-sentences such that for any L-structure M we have $M \models \tau$ if and only if the following all hold: $M \in \boldsymbol{H}(L)$, $R^M \in \mathscr{E}(M)$, $\mathrm{fld}(R^M) = U^M \neq \emptyset$, U^M/R^M is transitive, and $r(M) \leqq r$. Let $\boldsymbol{T}$ denote the class of L-structures in which τ is true. For any $M \in \boldsymbol{T}$ let $N(M)$ denote the quotient structure of M with universe U^M/R^M. Below we often write N for $N(M)$ for short.

Given arbitrary $M \in \boldsymbol{H}$ and transitive quotient structure N of M, we can expand M by adjoining N as a new unary relation, and then adjoining further relations so that the new structure will be in $\boldsymbol{H}$. Thus we make $L(M)$ like L, and N becomes $N(M)$. The expansion of M changes $l(M)$ and $r(M)$. However, their new values can be bounded in terms of the original ones. Hence to prove the theorem it suffices to find, given L as above and $r < \omega$, a number j such that, if $M \in \boldsymbol{T}$, and $|N| > j$, then N satisfies (ii) or (iii).

The idea of the proof is as follows. Clearly there is an L-sentence ξ such that $M \models \xi$ iff $\mathscr{E}(N)$ has a nontrivial member. Let κ_i be an L-sentence such that $M \models \kappa_i$ iff $|N| \geqq i$ $(i < \omega)$. We will show how to find an L-sentence π satisfying:

CLAIM. *For $M \in \boldsymbol{T}$ we have*

(a) *if N is an infinite perfect structure, then $M \models \pi$;*

(b) *if $M \models \pi$, then N is a perfect structure.*

Suppose the Claim is granted. From 9.3 and (a) of the Claim $\{\tau\} \cup \{\kappa_i : i < \omega\} \vdash \xi \vee \pi$. By the Compactness Theorem we can replace $\{\kappa_i : i < \omega\}$ by some finite subset. Hence there exists $j < \omega$ such that, if $M \in \boldsymbol{T}$ and $|N| > j$, then $M \models \xi$ or $M \models \pi$. Since ξ means that N satisfies (ii) of the conclusion, and, by (b) of the Claim, π implies that N satisfies (iii), we are done. It only remains to describe the sentence π and to prove the Claim.

From L we can compute k_0 such that if $M \in \boldsymbol{T}$ then $< k_0$ 2-types are realized in M. If N is perfect, then its width and index are $< k_0$. By 5.2 from L and r we can compute k_1 such that, if M' is obtained from $M \in \boldsymbol{T}$ by naming $< k_0$ equivalence classes of an equivalence relation in $E(M^2)$, then $l(M')$, $r(M') < k_1$. By 7.5 from L and r we can compute k_2 such that for such M', if $I \subseteq (M')^*$ is 0-definable and k_2-indiscernible and $[I] = 2$, then I is indiscernible in $(M')^*$. By 7.9 from L and r we can compute k_3 such that for such M', if $I_i \subseteq (M')^*$ $(i < j)$ are pairwise unlinked 0-definable indiscernible sets with $|I_i| \geqq k_3$ and $[I_i] = 2$, then the sets I_i $(i < j)$ are mutually indiscernible.

Let π be an L-sentence such that for $M \in \boldsymbol{T}$ we have $M \models \pi$ if and only if there exist $E_0, E_1 \in \mathscr{E}(N^2)$ such that the following conditions are satisfied:

(i) $E_1 \neq E_0$ and $\mathrm{fld}(E_1) = \mathrm{fld}(E_0)$,

(ii) N^2/E_0 is permuted transitively by aut(M),

(iii) $|N^2/E_0| < k_0$,

(iv) for all $C \in N^2/E_0$, the set C/E_1 is k_2-indiscernible in $(M, N^2/E_0)^*$,

(v) for all C_0, D_0, C_1, $D_1 \in N^2/E_1$ if $C_0/E_0 = D_0/E_0 \neq C_1/E_0 = D_1/E_0$ then $\langle C_0, C_1\rangle$ and $\langle D_0, D_1\rangle$ realize the same type in $(M, N^2/E_0)^*$,

(vi) $|C/E_1| > k_3$ for all $C \in N^2/E_0$,

(vii) there is a ternary relation 0-definable on N which induces a bijection between N and

$$B = \{X \subseteq N^2/E_1 : |X \cap C| = n \text{ for all } C \in N^2/E_0\}$$

for some $n < k_0$ such that $0 < 2n < |C/E_1|$ for $C \in N^2/E_0$.

Proof of Claim. (a) To see that $M \models \pi$ consider $E_0, E_1 \in \mathscr{E}(N^2)$ which together with a suitable ternary relation witness the perfectness of N. Then (v) is an easy consequence of the mutual indiscernibility of the classes C/E_1 $(C \in N^2/E_0)$. Since N is infinite so is $|C/E_1|$ for every $C \in N^2/E_0$, whence (vi) holds. All of the other clauses are clear.

(b) Let $M \in \boldsymbol{T}$ and $M \models \pi$. Let $E_0, E_1 \in \mathscr{E}(N^2)$ satisfy (i)-(vii) and M' denote $(M, N^2/E_0)$. Notice that $(M, N^2/E_0)$ can also be written (M, D) where $D = M^2/E_2$ for a certain $E_2 \in \mathscr{E}(M^2)$ and there is a natural bijection between N^2/E_0 and D which is 0-definable in M^*. Thus $l(M')$, $r(M') \leqq k_1$. From (ii) N^2/E_0 is permuted transitively. From (iv) for all $C \in N^2/E_0$ the set C/E_1 is k_2-indiscernible and hence indiscernible in M' by choice of k_2. From (v), if $C_0, C_1 \in N^2/E_0$ are distinct then C_0/E_1 and C_1/E_1 are unlinked in M'. From (vi), by choice of k_3 the sets

C/E_1 ($C \in N^2/E_0$) are mutually indiscernible in M' and hence in N. From (vii) there is a bijection between N and B which is 0-definable and such that $0 < 2n < |C/E_1|$ for $C \in N^2/E_0$. From all this it is clear that E_0 and E_1 witness the perfectness of M. This completes the proof of the claim.

10. Indiscernible formulas revisited

The main result of this section says that if I is a 0-definable indiscernible set in M^* where $M \in \boldsymbol{H}$, $J \subseteq I$, and $A \subseteq M$ such that $|I|$ is sufficiently large and $J \cap \mathrm{cl}^I(\{a\}) = \varnothing$ for all $a \in A$, then J is strongly indiscernible over A. As a preliminary we prove the following lemma which is important in its own right.

LEMMA 10.1. *There exists a function $F : \omega^5 \to \omega$ such that if $M \in \boldsymbol{H}$, $I \in S^*(\varnothing)$, $c, d \in M^*$, $I \subseteq M^*$ is indiscernible over $\{c\}$ and indiscernible over $\{d\}$, and*

$$|I| \geqq F(l(M), r(M), [I], [c], [d])$$

then I is indiscernible over $\{c, d\}$.

PROOF. We shall suppose that $|I|$ is very large compared with $l(M)$, $r(M)$, $[I]$, $[c]$, and $[d]$. We can suppose that $[d] = 1$. Until further notice by "class" we mean a class of one of the equivalence relations in $\mathscr{E}(M)$. We can suppose that d is a class and $d \subseteq D$ for some $D \in S(\varnothing)$. Towards a contradiction suppose I is not indiscernible over $\{c, d\}$. Fixing c and D, choose d maximal with respect to inclusion. Let $e \subseteq D$ be a minimal class properly including d and let $M_0 = (M, \{e\})$. From 5.2 we have bounds on $l(M_0)$ and $r(M_0)$. By maximality of d, I is indiscernible over $\{c\}$ in M_0. Also I is indiscernible over $\{d\}$ in M_0, because e is $\{d\}$-definable in M. Let E'_0 be the member of $\mathscr{E}(M)$ such that d is an E'_0-class. Let N denote the quotient structure of M whose universe is e/E'_0. By choice of e no nontrivial equivalence relation on N is 0-definable. Let cl_0 mean closure in I in M_0. From 7.4 we have a bound on $|\mathrm{cl}_0(\{c, d'\})|$ for $d' \in e/E'_0$. Since $\mathrm{cl}_0(\{c, d\}) \neq \varnothing$ we have

$$I = \bigcup \{\mathrm{cl}_0(\{c, d'\}) : d' \in e/E'_0\}$$

whence $|N|$ is very large.

From 9.4 N is perfect. Let $\langle E_0, E_1, R\rangle$ be a triple witnessing the perfectness of N in the sense of §9 and $\mathscr{F} = \{C/E_1 : C \in N^2/E_0\}$. Let $P = \bigcup \mathscr{F}$ and E denote the equivalence relation on P induced by E_0, then $P \in S^*(\varnothing)$. Both the width and index of N are bounded by $l(M)$ and so are small compared with $|I|$. Every element of P/E can be defined from a pair of elements of M. Thus there is a small subset of M from which every element of $\{c\} \cup (P/E)$ is definable. Therefore $\mathrm{cl}_0(\{c\} \cup (P/E))$ is small, whence $\mathrm{cl}_0(\{c\} \cup (P/E)) = \emptyset$ because $\mathrm{cl}_0(\{c\}) = \emptyset$ and P/E is 0-definable. Similarly $\mathrm{cl}_0(\{d\} \cup (P/E)) = \emptyset$.

Let $M_1 = (M_0, P/E)$. Until further notice we work in M_1. The E-classes are 0-definable mutually indiscernible sets, none of which is linked to I because I is indiscernible over $\{d\}$. Let

$$C^* = \bigcup\{\mathrm{cl}^C(\{c\}) : C \in P/E\}.$$

Since $|P/E|$ is small and 0-definable and the E-classes are large, $\mathrm{cl}^C(\{c\} \cup C^*) = \mathrm{cl}^C(\{c\})$ for each $C \in P/E$. Since C^* is small we can treat $\{c\} \cup C^*$ as though it were an element of M_1^*. By 7.7, if $C, C' \in P/E$ are distinct, then $C - \mathrm{cl}^C(\{c\})$ and I are unlinked over $\{c\} \cup C^*$ and $C - \mathrm{cl}^C(\{c\})$ and $C' - \mathrm{cl}^{C'}(\{c\})$ are unlinked over $\{c\} \cup C^*$. From 7.9, I and the sets $C - \mathrm{cl}^C(\{c\})$, $C \in P/E$ are mutually indiscernible over $\{c\} \cup C^*$.

Therefore, if $d_0, d_1 \in e/E_0'$ have the same components in $\mathrm{cl}^C(\{c\})$ for each $C \in P/E$, then d_0, d_1 realize the same type in M_1 over $\{c\} \cup I$. Hence the number of types into which e/E_0' splits in M_0 over $\{c\} \cup I$ is small. Further, $\mathrm{cl}_0(\{c, d'\})$ is small for any $d' \in e/E_0$. Thus $\bigcup\{\mathrm{cl}_0(\{c, d'\}) : d' \in e/E_0\}$ is small. But this set is $\{c\}$-definable in M_0, whence $\bigcup\{\mathrm{cl}_0(\{c, d'\}) : d' \in e/E_0\} = \emptyset$. In particular $\mathrm{cl}_0(\{c, d\}) = \emptyset$, so we are done.

COROLLARY 10.2. *There exists a function $F : \omega^5 \to \omega$ such that if $M \in \boldsymbol{H}$, $I \in S^*(\emptyset)$, $c, d \in M^*$, and*

$$|I| \geqq F(l(M), r(M), [I], [c], [d])$$

then $\mathrm{cl}^I(\{c, d\}) = \mathrm{cl}^I(\{c\}) \cup \mathrm{cl}^I(\{d\})$.

PROOF. Apply 10.1 to $(M, \mathrm{cl}^I(\{c\}) \cup \mathrm{cl}^I(\{d\}))$ and $I - (\mathrm{cl}^I(\{c\}) \cup \mathrm{cl}^I(\{d\}))$.

THEOREM 10.3. *There exists a function $F : \omega^3 \to \omega$ such that if $A \subseteq M \in \boldsymbol{H}$, $I \in S^*(\emptyset)$ is indiscernible, $J \subseteq I$, $\mathrm{cl}^I(\{a\}) \cap J = \emptyset$ for all $a \in A$, and $|I| \geqq F(l(M), r(M), [I])$ then J is strongly indiscernible over A.*

PROOF. Without loss we can suppose $[I] = 1$. We shall suppose that $|I|$ is very large compared with $l(M)$ and $r(M)$ and deduce the desired conclusion. From 7.3 there exists $d \in M^*$, $P \in S_1(\{d\})$, and $E \in \mathscr{E}(M)$ such that $I = \dot{M}/E$, $[d]$ is bounded in terms of $l(M)$ and $r(M)$, and if $a_1, \dots, a_k \in P$ are pairwise E-inequivalent, then $\mathrm{tp}(a_1, \dots, a_k \mid \{d\})$ depends only on k. Further, P/E is almost all of I, and if $C \subseteq M$ is defined from a small set of parameters then C splits at most a few of the sets $P \cap a$, where a runs through P/E. We require

CLAIM. *If* $a_1, \ldots, a_k \in P$ *are pairwise* E*-inequivalent and* $a_i/E \in J$ $(1 \leqq i \leqq k)$ *then* $\operatorname{tp}(a_1, \ldots, a_k \mid A \cup \{d\} \cup \operatorname{cl}^I(\{d\}))$ *depends only on* k.

PROOF OF CLAIM. Since $\operatorname{cl}^I(\{d\})$ is small $I(M)$ remains small if we name every member of $\{d\} \cup \operatorname{cl}^I(\{d\})$. Thus if the claim fails there exists a counterexample with both $|A|$ and k small. Consider such a counterexample and let C denote $A \cup \{d\} \cup \operatorname{cl}^I(\{d\})$. There exist $a_1, \ldots, a_k, a_{k+1} \in P$ pairwise E-inequivalent, except possibly for a_k and a_{k+1} such that $a_i/E \in J$ $(1 \leqq i \leqq k+1)$ and

$$\operatorname{tp}(a_1, \ldots, a_k \mid C) \neq \operatorname{tp}(a_1, \ldots, a_{k-1}, a_{k+1} \mid C).$$

Using 10.2 we have

$$\begin{aligned} \operatorname{cl}^I(C) &= \bigcup \{\operatorname{cl}^I(\{a\}) : a \in A\} \cup \operatorname{cl}^I(\{d\}) \cup \operatorname{cl}^I(\operatorname{cl}^I(\{d\})) \\ &= (I - P/E) \cup \bigcup \{\operatorname{cl}^I(\{a\}) : a \in A\} \end{aligned}$$

whence

$$\{a_i/E : 1 \leqq i \leqq k+1\} \cap \operatorname{cl}^I(C) = \varnothing.$$

By choice of d and P we have $\operatorname{cl}^I(\{d, a_i\}) = (I - P/E) \cup \{a_i/E\}$. Using 10.2 again we get

$$\operatorname{cl}^I(C \cup \{a_i : 1 \leqq i < k\}) = \operatorname{cl}^I(C) \cup \{a_i/E : 1 \leqq i < k\}.$$

Thus $I - (\operatorname{cl}^I(C \cup \{a_i/E : 1 \leqq i < k\}))$ is indiscernible over $C \cup \{a_i : 1 \leqq i < k\}$ which means that a_k, a_{k+1} can be chosen E-equivalent. It follows that some subset of M definable from $C \cup \{a_i : 1 \leqq i < k\}$ splits every set $P \cap a$, $a \in I - \operatorname{cl}^I(C)$. This contradicts one of the properties of P stated above, because $|C|$ and k are both small. This completes the proof of the claim.

From the claim $J \cap (P—E)$ is indiscernible over $A \cup (I - (P/E))$. But I is a 0-definable indiscernible set. Thus d can be chosen so that $I - (P/E)$ is any subset of I of the right cardinality. Since $|I|$ is large compared with $|I - (P/E)|$ it follows that J is indiscernible over A.

It remains to show that J is strongly indiscernible over A. Since M is either finite or $\aleph_0$-stable, I is strongly indiscernible by 3.4. Without loss of generality we can suppose that

$$A = \{a \in M : \operatorname{cl}^I(\{a\}) \cap J = \varnothing\} \quad \text{and} \quad J = I - \bigcup \{\operatorname{cl}^I(\{a\}) : a \in A\}.$$

Notice that $I - J \subseteq \operatorname{acl}_{M^*}(A)$ and that $a \in M$ belongs to A if it has the same type over I as an element of A. Let $M_0^* \preccurlyeq M^*$ be prime over $A \cup J$, then $I \subseteq M_0^*$.

Since M^* is $\aleph_0$-categorical, $M_0^* \simeq M^*$. Since I is strongly indiscernible there is an isomorphism $\alpha : M^* \to M_0^*$ with $I \subseteq \mathrm{fix}(\alpha)$. Clearly $\alpha(A) = A$ and so M^* is prime over $A \cup J$. By the uniqueness of prime models 3.2, J is strongly indiscernible over A.

11. Dissecting a structure

In this section we see how an arbitrary countable stable structure M homogeneous for a finite relational language can be partitioned by means of families of indiscernible sets canonically associated with the structure.

A family $\mathscr{F}$ of indiscernible sets in M^* is called a *pre-nice family attached to M* if there exist $E_0, E_1 \in \mathscr{E}(M^2)$ such that

(i) $\mathrm{fld}(E_0) = \mathrm{fld}(E_1)$ and $E_1 \subsetneqq E_0$,

(ii) $\mathscr{F} = \{C/E_1 : C \in M^2/E_0\}$,

(iii) for each $I \in \mathscr{F}$, I is indiscernible over $(\bigcup \mathscr{F}) - I$,

(iv) $\mathscr{F}$ is permuted transitively by $\mathrm{aut}(M)$.

Notice that (iii) says that $\mathscr{F}$ is a mutually indiscernible family. By 3.4 it follows that $\mathscr{F}$ is a strongly mutually indiscernible family.

A pair $\boldsymbol{\phi} = \langle \phi_0, \phi_1 \rangle$ of quantifier-free formulas of $L(M)$ having at most x_0, x_1, x_2, x_3 free is called a *pre-nice pair for M* if ϕ_0, ϕ_1 define E_0, E_1 on $\mathscr{E}(M^2)$ such that $\mathrm{fld}(E_0) = \mathrm{fld}(E_1)$, $E_1 \subsetneqq E_0$, and $\mathscr{F} = \{C/E_1 : C \in M^2/E_0\}$ is a pre-nice family attached to M.

Let $\mathscr{F}$ be a pre-nice family attached to M. From (iv) $\mathscr{F}$ is permuted transitively by $\mathrm{aut}(M)$, whence $|I|$ is the same for each $I \in \mathscr{F}$. This common cardinality is denoted $d^p(\mathscr{F})$ and is called the *dimension of* $\mathscr{F}$. If $\boldsymbol{\phi}$ is a pre-nice pair for M which defines $\mathscr{F}$ then we let $d^p_M(\boldsymbol{\phi}) = d^p(\mathscr{F})$ and we call $d^p_M(\boldsymbol{\phi})$ the $\boldsymbol{\phi}$*-dimension* of M.

If $\mathscr{F}$ is a pre-nice family attached to M and $A \subseteq M^*$ define $\mathrm{cl}^{\mathscr{F}}(A) = \bigcup\{\mathrm{cl}^I(A) : I \in \mathscr{F}\}$. Define $\mathrm{cl}^{\boldsymbol{\phi}}(A)$ where $\boldsymbol{\phi}$ is a pre-nice pair analogously. Note that $\mathrm{cl}^{\mathscr{F}}(A)$ is not defined for every $A \subseteq M^*$, because $\mathrm{cl}^I(A)$ is not. From 7.4 if $d^p(\mathscr{F})$ is sufficiently large compared with $l(M)$, $r(M)$, and $[a]$, then $\mathrm{cl}^{\mathscr{F}}(\{a\})$ exists. From the stability of M $\mathrm{cl}^{\mathscr{F}}(\{a\})$ is finite when it exists. Also, if $a \in M$ and $\mathrm{cl}^{\mathscr{F}}(\{a\})$ exists, then

$$|\mathrm{cl}^{\mathscr{F}}(\{a\})| \leqq |s_2(\varnothing)| \leqq l(M)$$

because as a_0, a_1 run through M, $|\mathrm{cl}^{\mathscr{F}}(\{a_0\}) \cap \mathrm{cl}^{\mathscr{F}}(\{a_1\})|$ takes all values $\leqq |\mathrm{cl}^{\mathscr{F}}(\{a\})|$. Let $\mathscr{F}$ be a pre-nice family attached to M. We call $\mathscr{F}$ a *nice family* if the following further conditions are satisfied:

(v) $\mathrm{cl}^{\mathscr{F}}(\{a\})$ exists for all $a \in M$ and $|\mathrm{cl}^{\mathscr{F}}(\{a\}) \cap I| < |I|/8$ for all $I \in \mathscr{F}$,

(vi) for all $a \in M$, if $\pi \in \mathrm{perm}(\bigcup \mathscr{F})$, $\pi(I) = I$ $(I \in \mathscr{F})$, and $\mathrm{cl}^{\mathscr{F}}(\{a\}) \subseteq \mathrm{fix}(\pi)$, then there exists $\alpha \in \mathrm{aut}(M)$ extending π with $\alpha(a) = a$,

(vii) if $J \subseteq I \in \mathscr{F}$ and $A = \{a \in M : \mathrm{cl}^{\mathscr{F}}(\{a\}) \cap J = \emptyset\}$ then J is strongly indiscernible over A.

In the context of stable homogeneous structures which is our main concern we can omit (vi) and restrict (vii) to the case $|J| = 2$ without the notion of nice family being affected. To see this, for any $I \in \mathscr{F}$ and $b_0, b_1 \in I$ let $\alpha(b_0, b_1) \in \mathrm{aut}(M)$ be such that $\alpha(b_i) = b_{1-i}$ $(i < 2)$ and $A(b_0, b_1) \subseteq \mathrm{fix}(\alpha(b_0, b_1))$, where

$$A(b_0, b_1) = \{a \in M : \mathrm{cl}^{\mathscr{F}}(\{a\}) \cap \{b_0, b_1\} = \emptyset\}.$$

Let $b \in (\bigcup \mathscr{F}) - \{b_0, b_1\}$. Since $\mathscr{F}$ is a mutually indiscernible family and $|\mathrm{cl}^{\mathscr{F}}(\{a\}) \cap I| < (|I| - 2)/2$ for all $a \in M$ there exist $a_0, a_1 \in A(b_0, b_1)$ such that $\langle a_0, a_1 \rangle$ represents b. We conclude that $b \in \mathrm{fix}(\alpha(b_0, b_1))$, whence $(\bigcup \mathscr{F}) - \{b_0, b_1\} \subseteq \mathrm{fix}(\alpha(b_0, b_1))$.

Let $\pi(b_0, b_1)$ denote the permutation of $\bigcup \mathscr{F}$ obtained by transposing b_0 and b_1 all other elements being fixed. Let $\pi \in \mathrm{perm}(\bigcup \mathscr{F})$ such that $\pi(I) = I$ $(I \in \mathscr{F})$. Choose a sequence $\langle \langle b_{0,i}, b_{1,i} \rangle : i < k \rangle$ with $k \leqq \omega$ such that for each $i \leqq k$, $b_{0,i}$, $b_{1,i} \in (\bigcup \mathscr{F}) - \mathrm{fix}(\pi)$ are distinct and belong to the same member of $\mathscr{F}$, and $\pi = \lim_{i<k} \pi_i$, where

$$\pi_i = \pi(b_{0,i}, b_{1,i})\pi(b_{0,i-1}, b_{1,i-1}) \cdots \pi(b_{0,0}, b_{1,0}).$$

If $k < \omega$ then by $\lim_{i<k} \pi_i$ we mean π_{k-1}. If $k = \omega$, $\pi = \lim_{i<k} \pi_i$ means that for all $b \in \bigcup \mathscr{F}$ $\pi_i(b) = \pi(b)$ for all sufficiently large $i < \omega$. Let

$$\alpha_i = \alpha(b_{0,i}, b_{1,i})\alpha(b_{0,i-1}, b_{1,i-1}) \cdots \alpha(b_{0,0}, b_{1,0}).$$

It it easy to see that $\lim_{i<k} \alpha_i$ exists. Denote it by α. Then $\alpha \in \mathrm{aut}(M)$, α extends π, and

$$\{a \in M : \mathrm{cl}^{\mathscr{F}}(\{a\}) \subseteq \mathrm{fix}(\pi)\} \subseteq \mathrm{fix}(\alpha).$$

Since π was an arbitrary permutation of $\bigcup \mathscr{F}$ inducing the identity on $\mathscr{F}$ we can immediately infer (vi) and (vii), although to construct α we only used the special case of (vii) in which $|J| = 2$.

The notion of *nice pair for M* is exactly parallel to that of nice family and has already been given in the Introduction.

Examples of nice families are provided by the perfect structures $M(F, m, n)$ from §9, where we need $m > 8n$ because of the requirement in (v) that $|\mathrm{cl}^{\mathscr{F}}(\{a\}) \cap I| < |I|/8$.

The importance of 10.3 is that it yields

LEMMA 11.1. *There exists $F:\omega^2\to\omega$ such that if $\mathcal{F}$ is a pre-nice family attached to $M\in\boldsymbol{H}$ and $d^p(\mathcal{F})\geqq F(l(M),r(M))$ then $\mathcal{F}$ is nice.*

REMARK. One could equally well formulate this lemma for a pre-nice pair ϕ. Henceforth we will assume that all our results for nice families apply to nice pairs and vice-versa.

PROOF. By 7.4, if $d^p(\mathcal{F})$ is large enough compared with $l(M)$ and $r(M)$ then for any $a\in M$ and $I\in\mathcal{F}$ $\mathrm{cl}^I(\{a\})$ exists. Hence $\mathrm{cl}^{\mathcal{F}}(\{a\})$ exists for all $a\in M$ and as remarked above $|\mathrm{cl}^{\mathcal{F}}(\{a\})|\leqq l(M)$. Thus condition (v) is satisfied if $d^p(\mathcal{F})$ is large enough. Let $I\in\mathcal{F}$. Apply 10.3 to the structure $(M,\{I\})$. If $d^p(\mathcal{F})=|I|$ is large enough compared with $l(M)$ and $r(M)$ then any $J\subseteq I$ is strongly indiscernible over $A=\{a\in M:\mathrm{cl}^I(\{a\})\cap J=\emptyset\}$. Thus (vii) is satisfied and we saw above that (vii) implies (vi). This completes the proof.

We record for future reference what we proved in showing that (vii) implies (vi).

LEMMA 11.2. *If $\mathcal{F}$ is a nice family attached to $M\in\boldsymbol{H}$ and $\pi\in\mathrm{perm}(\bigcup\mathcal{F})$ induces the identity on $\mathcal{F}$, then there exists $\alpha\in\mathrm{aut}(M)$ extending π such that*

$$\{a\in M:\mathrm{cl}^{\mathcal{F}}(\{a\})\subseteq\mathrm{fix}(\pi)\}\subseteq\mathrm{fix}(\alpha).$$

In the Introduction we defined $\Phi(M)$ to be the set of all nice pairs for M and $\approx_M$ to be the relation of equivalence for M on $\Phi(M)$. Let $\boldsymbol{F}(M)$ denote the set of all nice families attached to M. We now make some observations about nice families which follow easily from the definition and which will be used without special mention below.

FACT 1. If $\mathcal{F}\in\boldsymbol{F}(M)$, $a\in X\in\mathcal{F}$, and B is a minimal nonempty $\{a\}$-definable subset of $\mathcal{F}$, then there exists $\mathcal{G}\subseteq\mathcal{F}$ such that B is either $\{a\}\cup\bigcup\mathcal{G}$ or $(\bigcup\mathcal{G})-\{a\}$.

FACT 2. If $\mathcal{F}\in\boldsymbol{F}(M)$ and E_0, E_1 relate to $\mathcal{F}$ as in the definition of pre-nice family, then E_0 induces on $\bigcup\mathcal{F}$ the finest nonequality 0-definable equivalence relation on $\bigcup\mathcal{F}$. Equivalently, if $E\supsetneqq E_1$ is a 0-definable equivalence relation on $\mathrm{fld}(E_1)$, then $E\supseteq E_0$.

PROOF. Suppose R is a 0-definable equivalence relation on $\bigcup\mathcal{F}$ other than equality. Let $a\in X\in\mathcal{F}$ and Y be the R-class of a. Since R is not equality and $\bigcup\mathcal{F}$ is transitive, $\{a\}\subsetneqq Y$. By Fact 1, $Y\supseteq Z$ for some $Z\in\mathcal{F}$. For $c\in Z$ the

R-class of c includes the unique member of $\mathcal{F}$ containing c. By transitivity of $\bigcup \mathcal{F}$ the same is true for all $a \in \bigcup \mathcal{F}$. Therefore Y is a union of members of $\mathcal{F}$ as required.

FACT 3. Let $\mathcal{F}, \mathcal{G} \in \boldsymbol{F}(\boldsymbol{M})$ and $\alpha : \bigcup \mathcal{F} \to \bigcup \mathcal{G}$ be a 0-definable bijection. Then α induces a 0-definable bijection between $\mathcal{F}$ and $\mathcal{G}$.

PROOF. Let E_0, E_1 be the equivalence relations associated with $\mathcal{F}$ and E'_0, E'_1 be the equivalence relations associated with $\mathcal{G}$. From Fact 2, $\alpha^{-1}(E'_0) \supseteq E_0$ and $\alpha(E_0) \supseteq E'_0$. Hence $\alpha(E_0) = E'_0$ which is sufficient.

If $\mathcal{F}, \mathcal{G} \in \boldsymbol{F}(M)$, call $\mathcal{F}$, $\mathcal{G}$ *equivalent*, written $\mathcal{F} \approx \mathcal{G}$, if there is a 0-definable bijection between $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$. The relation $\approx$ on $\boldsymbol{F}(M)$ corresponds exactly to $\approx_M$ on $\Phi(M)$.

From the definition of nice family, if $M \in \boldsymbol{H}$ and $\mathcal{F} \in \boldsymbol{F}(M)$, then, for any $A \subseteq M$ with $|A| \leqq 4$, $\mathrm{cl}^{\mathcal{F}}(A)$ exists and

$$\mathrm{cl}^{\mathcal{F}}(A) = \bigcup\{\mathrm{cl}^{\mathcal{F}}(\{a\}) : a \in A\}.$$

Hence $\mathrm{cl}^{\mathcal{F}}(\{a\})$ exists for any $a \in M^*$ with $[a] \leqq 4$.

Call $\mathcal{F}$ *dependent on* $\mathcal{G}$, written $\mathcal{F} \leqq \mathcal{G}$, if for $a \in \bigcup \mathcal{F}$ $\mathrm{cl}^{\mathcal{G}}(\{a\})$ is nonempty. Let dependence be defined in an analogous way for nice pairs and let the dependence relation on $\Phi(M)$ be denoted $\leqq_M$. Call $\mathcal{F}$ *strictly dependent on* $\mathcal{G}$, written $\mathcal{F} < \mathcal{G}$, if $\mathcal{F} \leqq \mathcal{G}$ and $\mathcal{F} \not\approx \mathcal{G}$. Define strict dependence for nice pairs, denoted $<_M$, analogously.

Let $d_M : \Phi(M) \to \omega + 1$ denote the dimension function for M, i.e. $d^p_M \restriction \Phi(M)$. Let $d : \boldsymbol{F}(M) \to \omega + 1$ be $d^p \restriction \boldsymbol{F}(M)$.

LEMMA 11.3. *There exists $F : \omega \to \omega$ such that if $M \in \boldsymbol{H}$ and $\boldsymbol{F}'(M) = \{\mathcal{F} \in \boldsymbol{F}(M) : d(\mathcal{F}) \geqq F(l(M))\}$ then*

(i) *if $\mathcal{F}, \mathcal{G} \in \boldsymbol{F}(M)$, $\mathcal{H} \in \boldsymbol{F}'(M)$, $\mathcal{F} \leqq \mathcal{G}$, and $\mathcal{G} \leqq \mathcal{H}$, then $\mathcal{F} \leqq \mathcal{H}$,*

(ii) *if $\mathcal{F} \in \boldsymbol{F}'(M)$, $\mathcal{G} \in \boldsymbol{F}(M)$, $\mathcal{F} \leqq \mathcal{G}$, and $\mathcal{G} \leqq \mathcal{F}$, then $\mathcal{F} \approx \mathcal{G}$,*

(iii) *if $I \in \mathcal{F} \in \boldsymbol{F}(M)$ then I is strongly indiscernible over*

$$((\bigcup \mathcal{F}) - I) \cup \bigcup\{\bigcup \mathcal{G} : \mathcal{G} \in \boldsymbol{F}(M), \mathcal{G} \not\leqq \mathcal{F}\}.$$

PROOF. (i) Let $\mathcal{F}, \mathcal{G} \in \boldsymbol{F}(M)$ and $\mathcal{H} \in \boldsymbol{F}'(M)$ such that $\mathcal{F} \leqq \mathcal{G}$ and $\mathcal{G} \leqq \mathcal{H}$. Let $a \in \bigcup \mathcal{F}$. Define

$$A(a) = \bigcup\{\mathrm{cl}^{\mathcal{H}}(\{b\}) : b \in \mathrm{cl}^{\mathcal{G}}(\{a\})\},$$

then $A(a)$ is $\{a\}$-definable, $A(a) \subseteq \bigcup \mathcal{H}$, and $A(a) \neq \emptyset$. Since $[a] \leqq 2$, $\mathrm{cl}^{\mathcal{G}}(\{a\})$ has cardinality $\leqq |s_4(\emptyset)|$. Likewise $|\mathrm{cl}^{\mathcal{H}}(\{b\})| \leqq |s_4(\emptyset)|$ for $b \in \bigcup \mathcal{G}$. Thus F can be chosen so as to guarantee $d(\mathscr{H})>|A(a)|$. Then there exists $I\in\mathscr{H}$ such that $A(a)\cap I\neq\emptyset$ and $A(a)\not\supseteq I$, whence $\mathrm{cl}^I(\{a\})\neq\emptyset$. Therefore $\mathscr{F}\leqq\mathscr{H}$ and we are done.

(ii) Let $\mathscr{F}\in\boldsymbol{F}'(M)$ and $\mathscr{G}\in\boldsymbol{F}(M)$ such that $\mathscr{F}\leqq\mathscr{G}$ and $\mathscr{G}\leqq\mathscr{F}$. Let $a\in\bigcup\mathscr{F}$. Define

$$A(a)=\bigcup\{\mathrm{cl}^{\mathscr{F}}(\{b\}):b\in\mathrm{cl}^{\mathscr{G}}(\{a\})\}.$$

Choose F such that $d(\mathscr{F})>|A(a)|$. Then $A(a)$ is a nonempty $\{a\}$-definable subset of $\bigcup\mathscr{F}$ which splits some $I\in\mathscr{F}$. Hence $A(a)=\{a\}$, whence $\mathrm{cl}^{\mathscr{F}}(\{b\})=\{a\}$ for all $b\in\mathrm{cl}^{\mathscr{G}}(\{a\})$. Clearly $\mathrm{cl}^{\mathscr{G}}(\{a\})$ is $\{c\}$-definable for all $c\in\mathrm{cl}^{\mathscr{G}}(\{a\})$. Hence $|\mathrm{cl}^{\mathscr{G}}(\{a\})|=1$. Let $f:\bigcup\mathscr{F}\to\bigcup\mathscr{G}$ and $g:\bigcup\mathscr{G}\to\bigcup\mathscr{F}$ be the unique 0-definable maps such that $\mathrm{cl}^{\mathscr{G}}(\{a\})=\{f(a)\}$ $(a\in\bigcup\mathscr{F})$ and $\mathrm{cl}^{\mathscr{F}}(\{b\})=\{g(b)\}$ $(b\in\bigcup\mathscr{G})$. Then gf is the identity which implies $\mathscr{F}\approx\mathscr{G}$.

(iii) Let $I\in\mathscr{F}\in\boldsymbol{F}(M)$. Let $J\subseteq I$ and $|J|=2$. Then J is strongly indiscernible over $A=\{a\in M:\mathrm{cl}^{\mathscr{F}}(\{a\})\cap J=\emptyset\}$ by 10.3. Let $\mathscr{G}\in\boldsymbol{F}(M)$ such that $\mathscr{G}\not\leqq\mathscr{F}$. Let $b\in\bigcup\mathscr{G}$. Then $\mathrm{cl}^{\mathscr{F}}(\{b\})=\emptyset$ and so b has a representative $\langle b_0,b_1\rangle\in M^2$ such that $\mathrm{cl}^{\mathscr{F}}(\{b_i\})\cap J=\emptyset$ $(i<2)$. If $c\in(\bigcup\mathscr{F})-J$, $\mathrm{cl}^{\mathscr{F}}(\{c\})=\{c\}$ and so c has a representative $\langle c_0,c_1\rangle\in M^2$ such that $\mathrm{cl}^{\mathscr{F}}(\{c_i\})\cap J=\emptyset$ $(i<2)$. We conclude that there exists $\alpha\in\mathrm{aut}(M)$ transposing the members of J such that

$$((\bigcup\mathscr{F})-J)\cup\bigcup\{\bigcup\mathscr{G}:\mathscr{G}\in\boldsymbol{F}(M),\mathscr{G}\not\leqq\mathscr{F}\}\subseteq\mathrm{fix}(\alpha).$$

This is enough.

We now give an existence theorem for nice families which is an application of 9.4.

Lemma 11.4. *Given $G:\omega^3\to\omega$ there exists $H:\omega^2\to\omega$ such that, if $M\in\boldsymbol{H}$ and N is a quotient structure of M, then either $|N|\leqq H(l(M),r(M))$ or there exists $\mathscr{F}\in\boldsymbol{F}(N)$ such that $|\mathscr{F}|<\omega$ and $d(\mathscr{F})>G(l(M),r(M),|\mathscr{F}|)$.*

Proof. For notational convenience we assume that $N=M$. It is easy to modify the proof so that it applies to the general case. Let $G':\omega^3\to\omega$ be an auxiliary function to be specified later in terms of the given G. Let $M\in\boldsymbol{H}$ be given. We shall show that either we can find the desired $\mathscr{F}$ or $|M|$ can be bounded in terms of $l(M)$ and $r(M)$. Let $P\in S_1(\emptyset)$ have the greatest possible cardinality and $\langle E_i:i\leqq n\rangle$ be a sequence of maximal length in $\mathscr{E}(M)$ such that $\mathrm{fld}(E_i)=P$ $(i\leqq n)$ and $E_i\supsetneqq E_{i+1}$ $(i<n)$. (In general case let P be the 1-type of N with the greatest number of solutions and stipulate that E_0, E_n be such that $P=C/E_n$ for some E_0-class C.) Let C_i be an E_i-class and $n_i=|C_i/E_{i+1}|$ $(i<n)$.

Let $m_i = \Pi_{j<i} n_j$ $(i \leqq n)$ denote the number of E_i-classes. Either there exists i such that $n_i \leqq G'(l(M), r(M), m_i)$, or $|P|$ and hence $|M|$ can be bounded in terms of $l(M)$ and $r(M)$. Thus forgetting the previous E_i's we can suppose that we have $E_0 \supsetneqq E_1 \in \mathscr{E}(M)$ and an E_0-class C such that $\text{fld}(E_i) = P$ $(i < 2)$, $n = |C/E_1| \geqq G'(l(M), r(M), |P/E_0|)$, and there is no $E \in \mathscr{E}(M)$ such that $E_0 \supsetneqq E \supsetneqq E_1$.

Fix an E_0-class C, then $N' = C/E_1$ is a quotient structure of M and $\mathscr{E}(N')$ contains no nontrivial member. By 9.4, if $G'(l(M), r(M), m)$ is set large enough, then N' is perfect and so there is a nice family $\mathscr{F}'$ attached to N'. Let E'_0, E'_1 witness the perfectness of N', then $\mathscr{F}' = \{C'/E'_1 : C' \text{ is an } E'_0\text{-class}\}$. Let a denote $\langle a_0, a_1 \rangle \in M^2$ and similarly for a', b, and b'. Define $E^*_i \in \mathscr{E}(M^2)$ $(i < 2)$ by

$$a E^*_i a' \equiv \exists b\, \exists b' [b, b' \in C^2 \ \&\ \langle a, a' \rangle \text{ is conjugate to } \langle b, b' \rangle$$
$$\&\ \langle b_1/E'_1, b_1/E'_1 \rangle E_i \langle b'_0/E'_1, b'_1/E'_1 \rangle] \quad (a, a' \in M^2).$$

Let $\mathscr{F}^* = \{C_0/E^*_1 : C_0 \text{ is an } E^*_0\text{-class}\}$. Then $\mathscr{F}^*$ is a disjoint union of copies of $\mathscr{F}'$ one for each conjugate of C.

Any two members of $\mathscr{F}^*$ are conjugate and in fact conjugate to any member of $\mathscr{F}'$. Thus we may speak of "dim($\mathscr{F}^*$)" without ambiguity meaning $d(\mathscr{F}') = n$. Since $|\mathscr{F}'| = \text{width}(N') \leqq l(M)$ we have $|\mathscr{F}^*| \leqq |P/E_0| \cdot |\mathscr{F}'| \leqq |P/E_0| \cdot l(M)$. Thus by choosing $G'(l(M), r(M), |P/E_0|)$ sufficiently large we ensure that $\dim(\mathscr{F}^*) = n > G(l(M), r(M), i)$ $(i \leqq |\mathscr{F}^*|)$.

By coalescing members of $\mathscr{F}^*$ where appropriate we shall construct a pre-nice family $\mathscr{F}^-$ attached to M such that $|\mathscr{F}^-| \leqq |\mathscr{F}^*|$ and $d^p(\mathscr{F}^-) = \dim(\mathscr{F}^*)$. To this end we need the

Claim. *If I, $I' \in \mathscr{F}^*$ then* $\text{cl}^I(\{I'\}) = \emptyset$.

Proof. By 7.4, $\text{cl}^I(\{I'\})$ exists and its cardinality can be bounded in terms of $l(M)$. If $\text{cl}^I(\{I'\}) \neq \emptyset$, then $I = \bigcup \{\text{cl}^I(\{J\}) : J \in \mathscr{F}^*\}$. By appropriate choice of G' we have

$$|I| = n > |\mathscr{F}^*| \cdot \sup\{|\text{cl}^I(\{J\})| : J \in \mathscr{F}^*\},$$

whence the sets $\text{cl}^I(\{J\})$, $J \in \mathscr{F}^*$, cannot cover I. Therefore $\text{cl}^I(\{I'\}) = \emptyset$.

From the Claim if $I, I' \in \mathscr{F}^*$ then I, I' are 0-definable indiscernible sets in $(M, \{I, I'\})^*$ and so we may speak of I, I' being linked or unlinked. From 7.7, being linked is an equivalence relation on $\mathscr{F}^*$. Recall that definable bijections between linked sets are unique and hence the composition of the bijections between I_0 and I_1 and between I_1 and I_2 will be the bijection between I_0 and I_2.

Define $E_i^- \in \mathscr{E}(M^2)$ $(i<2)$ by

$$\mathrm{fld}(E_i^-) = \mathrm{fld}(E_i^*) \quad (i<2),$$

$$a\ E_0^-\ b \equiv [(a/E_0^*)/E_1^* \text{ and } (b/E_0^*)/E_1^* \text{ are linked}],$$

$$a\ E_1^-\ b \equiv [a\ E_0^-\ b, \text{ and } a/E_1^* \text{ and } b/E_1^* \text{ are mates in the definable 1–1}$$

$$\text{correspondence between } (a/E_0^*)/E_1^* \text{ and } (b/E_0^*)/E_1^*].$$

Let $\mathscr{F}^- = \{C_0/E_1^- : C_0 \text{ is an } E_0^-\text{-class}\}$. The sets in $\mathscr{F}^-$ are indiscernible and pairwise unlinked. Any two members of $\mathscr{F}^-$ are conjugate because the same is true of $\mathscr{F}^*$. By 7.9, $\mathscr{F}^-$ is a family of mutually indiscernible sets. The point here is that we can name every member of $\mathscr{F}^-$ — the sets remain indiscernible because "$\dim(\mathscr{F}^-)$" is large compared with $|\mathscr{F}^-|$, and pairwise unlinked by 7.4. Therefore $\mathscr{F}^-$ is a pre-nice family attached to M. We have also shown that $d^p(\mathscr{F}^-) > G(l(M), r(M), |\mathscr{F}^-|)$ is guaranteed by appropriate choice of G'. By 11.1 we can suppose that $\mathscr{F}^-$ is nice which completes the proof.

Let $\boldsymbol{F}^n(M)$ denote $\{\mathscr{F} \in \boldsymbol{F}(M) : d(\mathscr{F}) \geqq n\}$.

LEMMA 11.5. *Let $M \in \boldsymbol{H}$, $b \in M^*$, and $n < \omega$ such that $\mathrm{cl}^{\mathscr{F}}(\{b\})$ exists for all $\mathscr{F} \in \boldsymbol{F}^n(M)$. Let $B = \bigcup\{\mathrm{cl}^{\mathscr{F}}(\{b\}) : \mathscr{F} \in \boldsymbol{F}^n(M)\}$. Then the number of conjugates of b over B can be bounded in terms of $l(M)$, $r(M)$, n, and $[b]$.*

PROOF. Without loss of generality we can suppose that $n \geqq F(l(M))$ where F is from 11.3. Thus the conclusions of 11.3 apply to $\boldsymbol{F}^n(M)$ and will be used without special comment below. We proceed by induction on $m(b) = |\boldsymbol{G}(b)|$, where

$$\boldsymbol{G}(b) = \{\mathscr{F} \in \boldsymbol{F}^n(M) : \exists \mathscr{G} \in \boldsymbol{F}^n(M)[\mathscr{G} \leqq \mathscr{F} \ \&\ \mathrm{cl}^{\mathscr{G}}(\{b\}) \neq \varnothing]\}.$$

Let $b = a/E_a$, where $a \in M^k$, $k = [b]$, and $E_a \in \mathscr{E}(M^k)$. Let P be the set of all conjugates of a. For each $c \in P$ let $B(c) = \bigcup\{\mathrm{cl}^{\mathscr{F}}(\{c/E_a\}) : \mathscr{F} \in \boldsymbol{F}^n(M)\}$. Define $E \in \mathscr{E}(M^k)$ by

$$c_0\ E\ c_1 \equiv [c_0, c_1 \in P,\ B(c_0) = B(c_1), \text{ and } c_0/E_a,\ c_1/E_a \text{ are conjugate over } B(c_0)] \quad (c_0, c_1 \in M^k).$$

Let $C_a = a/E$ and $N_a = C_a/E_a$. Then N_a is a quotient structure of M^k whose elements are the conjugates of b over B. Supposing that $|N_a|$ is very large compared with $l(M)$, $r(M)$, n, and $[b]$ we shall deduce a contradiction.

By 11.4 we obtain $\mathscr{F}_a \in \boldsymbol{F}(N_a)$ such that $|\mathscr{F}_a| < \omega$ and $d(\mathscr{F}_a)$ is very large compared with $l(M)$, $r(M)$, n, $[b]$, and $|\mathscr{F}_a|$. Elements of $\bigcup \mathscr{F}_a$ are primarily represented by members of $(N_a)^2$ but can also be represented by members of $(C_a)^2$. Thus there are $E_0, E_1 \in \mathscr{E}((M^k)^2)$ such that $\mathrm{fld}(E_0) = \mathrm{fld}(E_1)$, for every E_0-class D there is a conjugate C of C_a such that $D \subseteq C^2$, and $E_0 \upharpoonright (C_a)^2$ and $E_1 \upharpoonright (C_a)^2$ define a family $\mathscr{F}'_a$ of mutually indiscernible sets such that there is a $\{C_a\}$-definable bijection between $\mathscr{F}_a$ and $\mathscr{F}'_a$. Let $\mathscr{F}'$ be the family defined by E_0, E_1.

Our aim is to construct a nice family $\mathscr{F}^*$ attached to M^k by coalescing linked members of $\mathscr{F}'$. Then we shall obtain a contradiction by showing that $\mathrm{cl}^{\mathscr{F}^*}(\{b\}) \neq \varnothing$ and $\mathscr{F}^*$ is equivalent to some $\mathscr{F} \in \boldsymbol{F}^n(M)$ but not to any $\mathscr{F} \in \boldsymbol{F}^n(M)$ such that $\mathrm{cl}^{\mathscr{F}}(\{b\}) \neq \varnothing$. The first step is

Claim 1. *If $I' \in \mathscr{F}'_a$, and $c \in I \in \mathscr{G} \in \boldsymbol{G}(b)$, then* $\mathrm{cl}^{I'}(\{c\}) = \varnothing$.

Proof. Towards a contradiction assume the hypothesis but that $\mathrm{cl}^{I'}(\{c\}) \neq \varnothing$. Fixing I' choose $\mathscr{G}$, I, and c so as to minimize $m(c)$. We claim that $\mathrm{cl}^{I'}(\{I\}) = \varnothing$. If not, let $D = \bigcup\{\mathrm{cl}^{\mathscr{F}}(\{I\}) : \mathscr{F} \in \boldsymbol{F}^n(M)\}$. Since $\mathrm{cl}^{\mathscr{F}}(\{I\}) \neq \varnothing$ and $\mathscr{F} \in \boldsymbol{F}^n(M)$ imply $\mathscr{G} < \mathscr{F}$ we have $\mathscr{G} \in \boldsymbol{G}(b) - \boldsymbol{G}(I)$, and so $m(I) < m(b)$. By the induction hypothesis the number of conjugates of I over D can be bounded in terms of $l(M)$, $r(M)$, and n. Since $I' = d(\mathscr{F}_a)$ is large it follows that $\mathrm{cl}^{I'}(D)$ exists and is nonempty, whence by 10.2, $\mathrm{cl}^{I'}(\{d\}) \neq \varnothing$ for some $d \in D$. There exist $\mathscr{G}'' \in \boldsymbol{F}^n(M)$ and $I'' \in \mathscr{G}''$ such that $\mathscr{G} < \mathscr{G}''$ and $d \in I''$. Since $\mathrm{cl}^{\mathscr{F}}(\{d\}) \neq \varnothing$ and $\mathscr{F} \in \boldsymbol{F}^n(M)$ imply $\mathscr{G}'' \leqq \mathscr{F}$, we have $\mathscr{G} \in \boldsymbol{G}(c) - \boldsymbol{G}(d)$, and so $m(d) < m(c)$. This contradicts the choice of c and completes the proof that $\mathrm{cl}^{I'}(\{I\}) = \varnothing$. It follows that the sets $\mathrm{cl}^{I'}(\{c'\})$ $(c' \in I)$ cover I'. Hence the sets $\mathrm{cl}^{I}(\{c'\})$ $(c' \in I')$ cover $I - \mathrm{cl}^{I}(\{I'\})$. Since N_a is transitive the sets $\mathrm{cl}^{I'}(\{b'\})$ $(b' \in N_a)$ cover I'. From all this it follows that the sets $\mathrm{cl}^{I}(\{b'\})$ $(b' \in N_a)$ cover I which contradicts the definition of N_a and so completes the proof of Claim 1.

In order to be able to define $\mathscr{F}^*$ we need

Claim 2. *If $I \in \mathscr{F}'_a$ and I' is a conjugate of I, then* $\mathrm{cl}^{I}(\{I'\}) = \varnothing$.

Proof. Recall that $\mathscr{F}'_a$ is B-definable. Choose B' such that $\langle I', B'\rangle$ is conjugate to $\langle I, B\rangle$. Towards a contradiction suppose $\mathrm{cl}^{I}(\{I'\}) \neq \varnothing$. The number of conjugates of I' over B' is $|\mathscr{F}'_a| = |\mathscr{F}_a|$ and $|I| = d(\mathscr{F}_a)$ is very large compared with $|\mathscr{F}_a|$. Hence $\mathrm{cl}^{I}(\{B'\})$ exists and is nonempty. By 10.2, $\mathrm{cl}^{I}(\{c\}) \neq \varnothing$ for some $c \in B'$. But there exists $\mathscr{G} \in \boldsymbol{F}^n(M)$ such that $c \in \bigcup \mathscr{G}$ and $\mathrm{cl}^{\mathscr{G}}(\{b\}) \neq \varnothing$. Thus we have a contradiction of Claim 1 which completes the proof.

Recall that $\mathscr{F}'$ is the union of all families conjugate to $\mathscr{F}'_a$. From Claim 2, if $I, I' \in \mathscr{F}'$, then I, I' are 0-definable indiscernible sets in $(M, \{I, I'\})^*$ and so are either linked or unlinked. Now we construct $\mathscr{F}^*$ from $\mathscr{F}'$ in exactly the same way we constructed $\mathscr{F}^-$ from $\mathscr{F}^*$ in the proof of 11.4. $\mathscr{F}^*$ is a family of indiscernible sets attached to M^k and defined by $E_0, E_1 \in \mathscr{E}((M^k)^2)$. Any $I, I' \in \mathscr{F}^*$ are conjugate and $\mathrm{cl}^I(\{I'\}) = \varnothing$. If $I \neq I'$, then I and I' are unlinked. By applying 10.3 to $I \in \mathscr{F}^*$ with $J \subseteq I$ a doubleton we see that the members of $\mathscr{F}^*$ are mutually indiscernible. Hence $\mathscr{F}^*$ is a pre-nice family attached to M^k. By 11.1 $\mathscr{F}^*$ is a nice family attached to M^k, because $d^p(\mathscr{F}^*) = d(\mathscr{F}_a)$ is very large compared with $l(M)$, $r(M)$, and k.

Since any member of $\bigcup \mathscr{F}_a$ is represented by a member of $(N)^2_a$ there exist $a_0, a_1 \in C_a$ such that $\langle a_0/E_a, a_1/E_a\rangle$ represents a member of $\bigcup \mathscr{F}_a$. Let $b_i = a_i/E_a$ $(i < 2)$. Then $\mathrm{cl}^{\mathscr{F}_a}(\{b_0, b_1\}) \neq \varnothing$, whence $\mathrm{cl}^{\mathscr{F}'_a}(\{b_0, b_1\}) \neq \varnothing$, whence $\mathrm{cl}^{\mathscr{F}^*}(\{b_0, b_1\}) \neq \varnothing$. By 10.2, $\mathrm{cl}^{\mathscr{F}^*}(\{b_i\}) \neq \varnothing$ for some $i < 2$. Since b_0, b_1 are conjugates of b we have $\mathrm{cl}^{\mathscr{F}^*}(\{b\}) \neq \varnothing$.

In the same way it is easy to see that there exists $c \in M$ such that $\mathrm{cl}^{\mathscr{F}^*}(\{c\}) \neq \varnothing$. For any $d \in \bigcup \mathscr{F}^*$ we can find $c_0, c_1 \in M$ such that $\{d\} = \mathrm{cl}^{\mathscr{F}^*}(\{c_0\}) \cap \mathrm{cl}^{\mathscr{F}^*}(\{c_1\})$. Therefore we can find $\mathscr{F} \in \boldsymbol{F}^n(M)$ equivalent to $\mathscr{F}^*$.

Finally, from Claim 1 it follows that no $\mathscr{F} \in \boldsymbol{F}^n(M)$ with $\mathrm{cl}^{\mathscr{F}}(\{b\}) \neq \varnothing$ is equivalent to $\mathscr{F}^*$. Were there such an $\mathscr{F}$, for any $c \in \bigcup \mathscr{F}$ we should have $I^* \in \mathscr{F}^*$ such that $\mathrm{cl}^{I^*}(\{c\}) \neq \varnothing$. Hence c and I^* could be chosen such that I^* is linked to $I' \in \mathscr{F}'_a$. Then $\mathrm{cl}^{I'}(\{c\}) \neq \varnothing$ and Claim 1 is contradicted by letting $\mathscr{G} = \mathscr{F}$. This completes the proof.

Lemma 11.6. *Let $M \in \boldsymbol{H}$, $\mathscr{F} \in \boldsymbol{F}(M)$, and $n < \omega$, such that $d(\mathscr{G}) < \omega$ for all $\mathscr{G} \in \boldsymbol{F}^n(M)$ with $\mathscr{F} < \mathscr{G}$. Then $|\mathscr{F}|$ can be bounded in terms of $l(M)$, $r(M)$, n, and*

$$\max\{d(\mathscr{G}) : \mathscr{G} \in \boldsymbol{F}^n(M), \mathscr{F} < \mathscr{G}\}.$$

Proof. Without loss of generality assume n is large enough so that $\leqq$ is transitive on $\boldsymbol{F}^n(M)$ and $\mathrm{cl}^{\mathscr{G}}(\{I\})$ exist for all $I \in \mathscr{F}$ and $G \in \boldsymbol{F}^n(M)$. We proceed by induction on $|\{\mathscr{G} \in F^n(M) : \mathscr{F} < \mathscr{G}\}|$ which is $\leqq l(M)$. We can suppose that a bound on $|\mathscr{G}|$ has already been computed for $\mathscr{G} \in \boldsymbol{F}^n(M)$ with $\mathscr{F} < \mathscr{G}$. Thus we have a bound for $|\bigcup \mathscr{G}|$. Let $I \in \mathscr{F}$ and

$$B = \bigcup\{\mathrm{cl}^{\mathscr{G}}(\{I\}) : \mathscr{G} \in \boldsymbol{F}^n(M), \mathscr{F} < \mathscr{G}\}.$$

Since $B \subseteq \bigcup\{\bigcup \mathscr{G} : \mathscr{G} \in \boldsymbol{F}^n(M), \mathscr{F} < \mathscr{G}\}$ we can bound the number of possibilities for B. Once B is fixed the number of possibilities for I is bounded by 11.5. This completes the proof.

Lemma 11.7. *Let $x = \langle x_0, x_1, x_2, x_3\rangle$, L be a finite relational language, and*

$\boldsymbol{\phi} = \langle \phi_0(x), \phi_1(x) \rangle$ be a pair of quantifier-free formulas. There is an L-sentence ψ such that for any $M \in \boldsymbol{H}(L)$, $\boldsymbol{\phi}$ is a nice pair for M iff $M \models \psi$.

PROOF. We can easily find an L-sentence θ saying that there exist $E_0, E_1 \in \mathscr{E}(M^2)$ such that ϕ_0, ϕ_1 define E_0, E_1, $\mathrm{fld}(E_0) = \mathrm{fld}(E_1)$, $E_1 \subsetneqq E_0$, and any two E_0-classes are conjugate.

Let $\mathscr{F}$ denote the family $\{C/E_1 : C \in M^2/E_0\}$. There is an L-sentence χ saying the following: there exists $B_a \subseteq \bigcup \mathscr{F}$ $(a \in M)$ uniformly $\{a\}$-definable such that

(i) $|B_a \cap I| < |I|/8$ for all $I \in \mathscr{F}$,

(ii) if $I \in \mathscr{F}$, $b_0, b_1 \in I$ are distinct, and

$$A = \{a \in M : B_a \cap \{b_0, b_1\} = \varnothing\},$$

then there exists $a_0, a_1, a_2, a_3 \in M^2$ such that $b_i = a_i/E_1 = a_{i+2}/E_1$ $(i < 2)$ and $\mathrm{tp}(a_0, a_1 \mid A) = \mathrm{tp}(a_3, a_2 \mid A)$, and

(iii) if $I \in \mathscr{F}$, $b_0, b_1 \in I$ are distinct, and $b \in (\bigcup \mathscr{F}) - \{b_0, b_1\}$ then there exist $a_0, a_1 \in M$ such that $b = \langle a_0, a_1 \rangle / E_1$ and $B_{a_i} \cap \{b_0, b_1\} = \varnothing$ $(i < 2)$.

Let $\psi = \theta \wedge \chi$.

Suppose $\boldsymbol{\phi}$ is a nice pair for $M \in \boldsymbol{H}(L)$. Then $M \models \psi$ because for B_a we can take $\mathrm{cl}^{\boldsymbol{\phi}}(\{a\})$.

Suppose $M \in \boldsymbol{H}(L)$ and that $M \models \psi$. Then ϕ_0, ϕ_1 define the family $\mathscr{F}$ attached to M. Let b_0, b_1, I, and A be as in (ii). From 3.3, M is prime over $A \cup \{a_i, a_{i+1}\}$ $(i = 0, 2)$. Since the mapping which fixes A point-wise and takes $\langle a_0, a_1 \rangle$ into $\langle a_3, a_2 \rangle$ is elementary, by 3.4 there exists $\alpha \in \mathrm{aut}(M)$ with $A \subseteq \mathrm{fix}(\alpha)$, $\alpha(a_0) = a_3$, and $\alpha(a_1) = a_2$. Thus $\{b_0, b_1\}$ is strongly indiscernible over A. From (iii), if $b \in (\bigcup \mathscr{F}) - \{b_0, b_1\}$ then b has a representative in A^2, i.e. $\{b_0, b_1\}$ is strongly indiscernible over $A \cup ((\bigcup \mathscr{F}) - \{b_0, b_1\})$.

It follows easily that if $I \in \mathscr{F}$ then $I - B_a$ is strongly indiscernible over $\{a\} \cup B_a \cup ((\bigcup \mathscr{F}) - I)$. Hence $\mathrm{cl}^{\boldsymbol{\phi}}(\{a\})$ exists for all $a \in M$. It is now apparent that conditions (i)–(v) in the definition of nice family are satisfied and that (vii) holds when $|J| = 2$. From the remark following the definition of nice family $\mathscr{F} \in \boldsymbol{F}(M)$. Hence $\boldsymbol{\phi} \in \Phi(M)$. This completes the proof.

In conclusion we echo our opening remark. Fix $n < \omega$. With each $a \in M$ associate $B^n(a) = \bigcup\{\mathrm{cl}^{\mathscr{F}}(\{a\}) : \mathscr{F} \in \boldsymbol{F}^n(M)\}$. Then $|B^n(a)|$ is bounded in terms of $l(M)$. Define $E^n \in \mathscr{E}(M)$ by

$$a_0\, E^n\, a_1 \equiv [B^n(a_0) = B^n(a_1)\ \&\ \mathrm{tp}(a_0 \mid B^n(a_0)) = \mathrm{tp}(a_1 \mid B^n(a_1))].$$

Then E^n is 0-definable and classifies elements of M according to their closures in the members of $\boldsymbol{F}^n(M)$. By 11.5 the cardinality of the E^n-classes can be bounded in terms of $l(M)$, $r(M)$, and n. We have introduced the parameter n because sometimes it is convenient to exclude from consideration nice families of small dimension.

12. Shrinking a structure

In the last section we studied nice pairs and the associated nice families of indiscernible sets. Here we shall investigate in detail the notion of shrinking which was touched upon briefly in the Introduction.

First we establish some notation. Let $M \in \boldsymbol{H}$, $\boldsymbol{\phi} \in \Phi(M)$, and $m < d_M(\boldsymbol{\phi})$. Let $\mathcal{F} \in \boldsymbol{F}(M)$ be the nice family associated with $\boldsymbol{\phi}$. Choose $B \subseteq \bigcup \mathcal{F}$ such that $|B \cap I| = m$ for all $I \in \mathcal{F}$. Let $M(\boldsymbol{\phi}, m)$ (also denoted $M(\mathcal{F}, m)$) be the substructure of M (i.e. the $L(M)$-substructure) whose universe is $\{a \in M : \mathrm{cl}^{\boldsymbol{\phi}}(\{a\}) \subseteq B\}$. If there is no $a \in M$ such that $\mathrm{cl}^{\boldsymbol{\phi}}(\{a\}) \subseteq B$, then $M(\boldsymbol{\phi}, m)$ is undefined. Since $\mathcal{F}$ is a strongly mutually indiscernible family any two possibilities for $M(\boldsymbol{\phi}, m)$ are conjugate regarded as subsets of M.

The structure $M(\boldsymbol{\phi}, m)$ is said to be obtained from M by a *one-step shrinking*; m is called the *target dimension.*

A substructure $N \subseteq M$ is called *homogeneous* if it is full and for every $\alpha \in \mathrm{aut}(M)$ and finite $A \subseteq N$ such that $\alpha(A) \subseteq N$ there exists $\beta \in \mathrm{aut}(M)$ such that $\beta \restriction A = \alpha \restriction A$ and $\beta(N) = N$.

Observe that, if N is a homogeneous substructure of $M \in \boldsymbol{H}$, then N is homogeneous for $L(M)$. But the converse is not generally true when N is infinite.

LEMMA 12.1. *Let $M \in \boldsymbol{H}$, $\boldsymbol{\phi} \in \Phi(M)$, $\mathcal{F} \in \boldsymbol{F}(M)$ be the associated nice family, $m < d_M(\boldsymbol{\phi})$, and $M' = M(\boldsymbol{\phi}, m)$ exist. Then M' has the following properties:*

(i) *M' is a homogeneous substructure of M.*

(ii) *M is prime over M'.*

(iii) *Two tuples in M' are conjugate in M' iff they are conjugate in M.*

(iv) *Every automorphism of M' extends to an automorphism of M.*

(v) *If $\bar{a} \in M$, then $\mathrm{tp}(\bar{a})$ is realized in M' iff*

$$|\bigcup\{\mathrm{cl}^I(\{a\}) : a \in \mathrm{rng}(\bar{a})\}| \leqq m \qquad (I \in \mathcal{F}).$$

(vi) *If $a \in M^*$ is an equivalence class of k-tuples from M and m is large enough compared with k and $l(M)$, then $a \in (M')^*$ (more precisely, a has a representative in M') iff $\mathrm{cl}^{\mathcal{F}}(\{a\}) \subseteq B$.*

PROOF. (i) Let $A \subseteq M'$, $|A| < \omega$, $\alpha \in \operatorname{aut}(M)$, and $\alpha(A) \subseteq M'$. Let $C = \bigcup\{\mathrm{cl}^{\phi}(\{a\}) : a \in A\}$. Then $C \subseteq B$, $|C| < \omega$, and $\alpha(C) \subseteq B$. By considering each $I \in \mathcal{F}$ separately we can find $\sigma \in \operatorname{perm}(\bigcup \mathcal{F})$ such that $\sigma(I) = I$ $(I \in \mathcal{F})$, $\sigma \restriction \alpha(C)$ is the identity, and $[\sigma(\alpha \restriction \bigcup \mathcal{F})](B) = B$. By 11.2 there exists $\gamma \in \operatorname{aut}(M)$ extending σ with $\{a \in M : \mathrm{cl}^{\phi}(\{a\}) \subseteq \alpha(C)\} \subseteq \operatorname{fix}(\gamma)$. If $a \in A$, then $\{\mathrm{cl}^{\phi}(\{\alpha(a)\}) = \alpha(\mathrm{cl}^{\phi}(\{a\})) \subseteq \alpha(C)$. Hence $\alpha(A) \subseteq \operatorname{fix}(\gamma)$ and $(\gamma\alpha)(B) = B$. Let $\beta = \gamma\alpha \in \operatorname{aut}(M)$. Then $\beta \restriction A = \alpha \restriction A$ and $\beta(B) = B$. From $\beta(B) = B$ it is immediate that $\beta(M') = M'$, so we are done.

(ii) Regard M' as a subset of M. Let N be a model of $\operatorname{Th}(M)$ prime over M'. Since $M \in \boldsymbol{C}$ there exists an isomorphism $\alpha : M \to N$. Let $\mathcal{G} = \alpha(\mathcal{F})$ be the nice family associated with $\boldsymbol{\phi}$ attached to N. By construction of M', for each $I \in \mathcal{F}$, $\bigcup\{\mathrm{cl}^{I}(\{a\}) : a \in M'\} = I \cap B$ has cardinality m. Since N can be found as an elementary submodel of M, $|\bigcup\{\mathrm{cl}^{J}(\{a\}) : a \in M'\}| = m$ for each $J \in \mathcal{G}$. Let $C = \bigcup\{\mathrm{cl}^{J}(\{a\}) : a \in M', J \in \mathcal{G}\}$. If $a \in N - M'$ then $\mathrm{cl}^{\mathcal{G}}(\{a\}) \not\subseteq C$ again because N can be found as an elementary submodel of M. Using C to construct $N(\boldsymbol{\phi}, m)$ we have $N(\boldsymbol{\phi}, m) = M'$. Since N is prime over $M' = N(\boldsymbol{\phi}, m)$ and M, N are isomorphic, we are done.

(iii) This is an obvious consequence of (i).

(iv) This is immediate from (ii) and 3.2.

(v) Let $\bar{a} \in M$ and $C = \bigcup\{\mathrm{cl}^{\mathcal{F}}(\{a\}) : a \in \operatorname{rng}(\bar{a})\}$. If $|C \cap I| \leqq m$ for all $I \in \mathcal{F}$ then B for the construction of M' can be chosen $\supseteq C$. Hence M' realizes $\operatorname{tp}(\bar{a})$. If $|C \cap I| > m$ for some $I \in \mathcal{F}$, then for any $\bar{d} \in M$ conjugate to $\bar{a}$ there exists $J \in \mathcal{F}$ such that $|\bigcup\{\mathrm{cl}^{J}(\{a\}) : a \in \operatorname{rng}(\bar{d})\}| > m$. Thus, for any $\bar{d} \in M$ conjugate to $\bar{a}, \bar{d} \in M'$. This is enough.

(vi) Let $\bar{b} \in M^k$ be a representative of $a \in M^*$ and suppose m is large compared with $l(M)$ and k. Clearly, $\mathrm{cl}^{\mathcal{F}}(\{a\}) \subseteq \mathrm{cl}^{\mathcal{F}}(\operatorname{rng}(\bar{b}))$. If $\bar{b} \in M'$, by 10.2 we have $\mathrm{cl}^{\mathcal{F}}(\operatorname{rng}(\bar{b})) \subseteq B$, whence $\mathrm{cl}^{\mathcal{F}}(\{a\}) \subseteq B$. Now suppose $\mathrm{cl}^{\mathcal{F}}(\{a\}) \subseteq B$. Since m is large compared with $l(M)$ and k, $|\mathrm{cl}^{\mathcal{F}}(\operatorname{rng}(\bar{b}))| \leqq m$ by the argument used to bound $\mathrm{cl}^{I}(\{a\})$ in 7.4. Since $\mathcal{F}$ is a nice family, there exists $\alpha \in \operatorname{aut}(M)$ such that

$$\alpha(\mathrm{cl}^{\mathcal{F}}(\operatorname{rng}(\bar{b}))) \subseteq B \quad \text{and} \quad \{a\} \cup \mathrm{cl}^{\mathcal{F}}(\{a\}) \subseteq \operatorname{fix}(\alpha).$$

Then $\alpha(\bar{b}) \in M'$ is a representative of a.

LEMMA 12.2. *Let $M \in \boldsymbol{H}$, $\boldsymbol{\phi} \in \Phi(M)$, $m < d_M(\boldsymbol{\phi})$, and $M' = M(\boldsymbol{\phi}, m)$ exist. Then M' has the following properties*:

(i) *Let $\theta(\bar{x})$ be a formula of $L(M)$ and m be large enough compared with $l(M)$ and $l(\theta(\bar{x}))$ the length of $\theta(\bar{x})$. For all $\bar{a} \in M'$ with $l(\bar{a}) = l(\bar{x})$, $M \models \theta(\bar{a})$ iff $M' \models \theta(\bar{a})$.*

(ii) *If* m *is large enough compared with* $l(M)$, *then* $l(M)=l(M')$, $\Phi(M)=\Phi(M')$, $\approx_M=\approx_{M'}$ *and* $\leqq_M=\leqq_{M'}$.

(iii) $r(M')\leqq r(M)$ *and equality holds whenever* m *is large enough compared with* $l(M)$ *and* $r(M')$.

Proof. (i) We proceed by induction on the formula $\theta(\bar{x})$. The only nontrivial case is that in which $\theta(\bar{x})$ is $\exists y\psi(\bar{x},y)$. Let $\bar{a}\in M'$ and suppose $M\models\theta(\bar{a})$. Then there exists $b\in M$ such that $M\models\psi(\bar{a},b)$. Provided m is large enough compared with $l(M)$ and $l(\bar{x})$, $|\mathrm{cl}^{\mathscr{F}}(\{b\}\cup\mathrm{rng}(\bar{a}))|\leqq m$. Since $\mathscr{F}$ is a nice family, there exists $\alpha\in\mathrm{aut}(M)$, such that

$$\alpha(\mathrm{cl}^{\mathscr{F}}(\{b\}))\subseteq B\quad\text{and}\quad\{\bar{a}\}\cup\mathrm{cl}^{\mathscr{F}}(\{\bar{a}\})\subseteq\mathrm{fix}(\alpha).$$

Now $\alpha(b)\in M'$ and $M\models\psi(\bar{a},\alpha(b))$. By the induction hypothesis $M'\models\psi(\bar{a},\alpha(b))$. Hence $M'\models\theta(\bar{a})$. Using only the induction hypothesis, we see that, if $M'\models\theta(\bar{a})$ is given, then $M\models\theta(\bar{a})$.

(ii) This follows from (i). The only tricky part is showing that "$\boldsymbol{\phi}\in\Phi(M)$" can be expressed by an $L(M)$-sentence which was done in 11.7.

(iii) Looking back at the definition of rank in §4 one sees that $r(M)=r>0$ means that a certain tree of 1-types of height r exists. To verify such a tree we need a witness for each of the 2^r branches and a subset of M of cardinality at most $a(M)-1$ for each of the 2^r-1 nodes. Thus the rank of M is witnessed by a subset A of M of cardinality at most

$$(a(M)-1)\cdot(2^{r(M)}-1)+2^{r(M)}<a(M)\cdot 2^{r(M)}.$$

The subset A' of M' witnessing the rank of M' witnesses that M has rank $\geqq r(M')$. Also, if $r(M)>r(M')$ and m is large enough compared with $l(M)$ and $r(M')$, then M' includes some subset which witnesses that $r(M')>r(M')$, contradiction.

Before stating the next lemma we need further terminology. Let M, $\boldsymbol{\phi}$, m, and M' be as in the previous lemmas. Let $\boldsymbol{\psi}\in\Phi(M)\cap\Phi(M')$ and $\mathscr{G}\in\boldsymbol{F}(M)$, $\mathscr{G}'\in\boldsymbol{F}(M')$ be the nice families attached to M, M' respectively which are associated with $\boldsymbol{\psi}$. The injection of M' into M gives rise to obvious *canonical embeddings* of $\bigcup\mathscr{G}'$ into $\bigcup\mathscr{G}$ and of $\mathscr{G}'$ into $\mathscr{G}$. If $b\in\bigcup\mathscr{G}$ is the image of $b'\in\bigcup\mathscr{G}'$ under the canonical embedding then we say that b and b' *correspond*. Similarly, if $I\in\mathscr{G}$ is the image of $I'\in\mathscr{G}'$ under the canonical embedding we say that I and I' *correspond*. The canonical embedding of $\bigcup\mathscr{G}'$ into $\bigcup\mathscr{G}$ restricts to a *canonical embedding* of I' into I.

LEMMA 12.3. *Let M, ϕ, m, and M' be as in* 12.1, 12.2. *Let $\psi \in \Phi(M) \cap \Phi(M')$ and $\mathscr{G} \in \boldsymbol{F}(M)$, $\mathscr{G}' \in \boldsymbol{F}(M')$ be associated with ψ. Let $I \in \mathscr{G}$ correspond to $I' \in \mathscr{G}'$. Provided that m is sufficiently large compared with $l(M)$ the following relationships hold between M and M':*

(i) *The canonical embedding of $\bigcup \mathscr{G}'$ into $\bigcup \mathscr{G}$ is a bijection iff $\psi \not\leqq_M \phi$.*

(ii) *The canonical embedding of $\mathscr{G}'$ into $\mathscr{G}$ is a bijection iff $\psi \not\leqq_M \phi$.*

(iii) *If $\psi \neq \phi$ the canonical embedding of I' into I is a bijection and $d_{M'}(\psi) = d_M(\psi)$.*

(iv) *If $\psi = \phi$ the canonical embedding of I' into I has range $B \cap I$ where $B \subseteq \bigcup \mathscr{F}$ is the set chosen in constructing M'.*

(v) *If $\psi \approx \phi$ then $d_{M'}(\psi) = m$.*

(vi) *If $a \in M'$, then $\mathrm{cl}^{\psi}_M(\{a\})$ is the image of $\mathrm{cl}^{\psi}_{M'}(\{a\})$ under the canonical embedding of $\bigcup \mathscr{G}'$ into $\bigcup \mathscr{G}$.*

PROOF. Let $\mathscr{F} \in \boldsymbol{F}(M)$, $\mathscr{F}' \in \boldsymbol{F}(M')$ be the nice families associated with ϕ. We assume that m is large compared with $l(M)$ so that 11.3 (i), (ii) and 12.2 (i), (ii) are available.

(i) Suppose $\psi \leqq_M \phi$, i.e. $\mathscr{G} \leqq \mathscr{F}$. Then $\mathrm{cl}^{\mathscr{F}}(\{b\}) \neq \varnothing$ for all $b \in \bigcup \mathscr{G}$. Choose b such that $\mathrm{cl}^{\mathscr{F}}(\{b\}) \subseteq B$, then b has no representative in $(M')^2$, whence no $b' \in \bigcup \mathscr{G}'$ corresponds to b. Now suppose $\mathscr{G} \not\leqq \mathscr{F}$. For any $b \in \bigcup \mathscr{G}$, $\mathrm{cl}^{\mathscr{F}}(\{b\}) = \varnothing$. Therefore b has a representative $\langle a_0, a_1 \rangle \in M^2$ such that $\mathrm{cl}^{\mathscr{F}}(\{a_i\}) \subseteq B$ $(i < 2)$. Thus $a_i \in M'$ $(i < 2)$. The element b' of $\bigcup \mathscr{G}'$ represented by $\langle a_0, a_1 \rangle$ corresponds to b. Hence in this case the canonical embedding of $\bigcup \mathscr{G}'$ into $\bigcup \mathscr{G}$ is a bijection.

(ii) Suppose $\psi <_M \phi$, i.e. $\mathscr{G} < \mathscr{F}$. Consider $b \in I$. We have $\mathrm{cl}^{\mathscr{F}}(\{b\}) \neq \varnothing$. Let $J \in \mathscr{F}$ be such that $\mathrm{cl}^J(\{b\}) \neq \varnothing$. From 11.3 (ii) $\mathscr{F} \not\leqq \mathscr{G}$ and from 11.3 (iii) I is strongly indiscernible over $\bigcup \mathscr{F}$, whence $\mathrm{cl}^I(\{J\}) = \varnothing$. If $\mathrm{cl}^J(\{I\}) = \varnothing$, then I and J are linked in $(M, \{I, J\})^*$ which contradicts $\mathscr{F} \not\leqq \mathscr{G}$. Therefore $\mathrm{cl}^J(\{I\}) \neq \varnothing$ and we can choose $I_0 \in \mathscr{G}$ such that $\mathrm{cl}^{\mathscr{F}}(\{I_0\}) \not\subseteq B$. Then I_0 has no mate in $\mathscr{G}'$, i.e. the embedding of $\mathscr{G}'$ into $\mathscr{G}$ is not onto.

Suppose $\psi \approx_M \phi$, i.e. $\mathscr{G} \approx \mathscr{F}$. Then $\mathrm{cl}^{\mathscr{F}}(\{J\}) = \varnothing$ for all $J \in \mathscr{G}$, whence every $J \in \mathscr{G}$ has a mate in $\mathscr{G}'$.

Suppose $\psi \not\leqq_M \phi$. Then the embedding of $\mathscr{G}'$ into $\mathscr{G}$ is a bijection by (i).

(iii) This is clear from (i) if $\psi \not\leqq_M \phi$. Thus suppose $\psi \leqq_M \phi$. From 11.3 (ii), $\mathscr{F} \not\leqq \mathscr{G}$ and from 11.3 (iii), I is strongly indiscernible over $\bigcup \mathscr{F}$, whence every $b \in I$ has a mate $b' \in I'$. This is enough.

(iv), (v). These are clear from the construction of M'.

(vi) Let $a \in M'$, and $a_0, a_1 \in (M')^2$ represent $b_0, b_1 \in I$ and $b'_0, b'_1 \in I'$. We can construct a formula $\theta(x_0, \ldots, x_4)$ of $L(M)$ such that, in M, $\theta(a, a_0, a_1)$ means that $\mathrm{tp}_M(a, b_0) = \mathrm{tp}_M(a, b_1)$. By 12.2 (i), $M \models \theta(a, a_0, a_1)$ iff $M' \models \theta(a, a_0, a_1)$. Thus b_0, b_1 realize different types over $\{a\}$ in M' if b_0', b_1' realize different types over $\{a\}$ in M. Now $\mathrm{cl}^I(\{a\})$ is the union of the "small" classes when I is partitioned by the equivalence relation of realizing the same type over $\{a\}$. Therefore $\mathrm{cl}^{I'}(\{a\})$ is mapped into $\mathrm{cl}^I(\{a\})$ under the canonical embedding of I' into I. Also there is a sentence fixing the number of "small" classes in the partition of I by $\{a\}$ and the cardinalities of these classes. Hence $\mathrm{cl}^{I'}(\{a\})$ is taken onto $\mathrm{cl}^I(\{a\})$ by the canonical embedding.

Let $n = |\mathrm{cl}^{\mathscr{G}}(\{a\})|$. Recall that n is bounded in terms of $l(M)$. There is a formula $\theta(x)$ of $L(M)$ such that, in M, $\theta(a)$ means $n = |\mathrm{cl}^{\mathscr{G}}(\{a\})|$. By 12.2 (i), $M' \models \theta(a)$ since $M \models \theta(a)$. Hence $|\mathrm{cl}^{\mathscr{G}'}(\{a\})| = n = |\mathrm{cl}^{\mathscr{G}}(\{a\})|$. Since $\mathrm{cl}^{\mathscr{G}}(\{a\}) = \bigcup\{\mathrm{cl}^I(\{a\}) : I \in \mathscr{G}\}$ and similarly for G' and I', $\mathrm{cl}^{\mathscr{G}'}(\{a\})$ is mapped onto $\mathrm{cl}^{\mathscr{G}}(\{a\})$ by the canonical embedding.

We now investigate the effect of iterating one-step shrinking. We first consider the case in which both one-step shrinkings are with respect to the same nice pair.

Lemma 12.4. *Let $M, M' \in \boldsymbol{H}$, $\phi \in \Phi(M)$, $m' < m < d_M(\phi)$, and $M' = M(\phi, m)$. If m is large enough compared with $l(M)$, then $\phi \in \Phi(M')$, $M(\phi, m')$ exists iff $M'(\phi, m')$ exists, and if both exist then they are conjugate as subsets of M.*

Proof. From 12.2 (ii) $\phi \in \Phi(M')$. Let $\mathscr{F} \in \boldsymbol{F}(M)$, $\mathscr{F}' \in \boldsymbol{F}(M')$ be associated with ϕ. Let $B, B' \subseteq \bigcup \mathscr{F}$ be the sets chosen in the construction of $M', M(\phi, m')$. Choose $B' \subseteq B$. From 12.3 (ii), (iv) there is unique $B'' \subseteq \bigcup \mathscr{F}'$ mapped onto B' by the canonical embedding of $\bigcup \mathscr{F}'$ into $\bigcup \mathscr{F}$. Since $|B'' \cap I| = m'$ for all $I \in \mathscr{F}'$ we can use B'' to construct $M'(\phi, m')$. Certainly $M(\phi, m') \subseteq M'$. Consider $a \in M'$. By 12.3 (vi), $\mathrm{cl}^{\phi}_{M'}(\{a\})$ is mapped onto $\mathrm{cl}^{\phi}_M(\{a\})$ under the canonical embedding of $\bigcup \mathscr{F}'$ into $\bigcup \mathscr{F}$. Hence $a \in M'(\phi, m')$ iff $a \in M(\phi, m')$. Thus for the given choice of B, B', B'' the structures $M(\phi, m')$ and $M'(\phi, m')$ are actually equal. From 12.1 (iv) any automorphism of M' extends to an automorphism of M. It follows that any two possibilities for $M'(\phi, m')$ are conjugate in M.

If ϕ, ψ are nice pairs for M which are equivalent, then a one-step shrinking with respect to ϕ is the same as a one-step shrinking with respect to ψ. We now consider inequivalent nice pairs ϕ and ψ and the effect of shrinking first with respect to one and then with respect to the other.

Lemma 12.5. *Let $M \in \boldsymbol{H}$, ϕ, $\psi \in \Phi(M)$, $\phi \not\sim_M \psi$, $m < d_M(\phi)$, and $n < d_M(\psi)$. If m, n are large enough compared with $l(M)$ then $(M(\phi, m))(\psi, n)$ and $(M(\psi, n))(\phi, m)$ both exist and are conjugate as subsets of M.*

REMARK. Intuitively the lemma says that provided the "target" dimensions m, n are large enough compared with $l(M)$ the order in which we perform the one-step shrinkings does not matter.

PROOF. Let $\mathcal{F}, \mathcal{G} \in \boldsymbol{F}(M)$ be associated with ϕ, ψ respectively. Let $\mathcal{G}' \in \boldsymbol{F}(M(\phi, m))$, $\mathcal{F}' \in \boldsymbol{F}(M(\psi, n))$ be associated with ψ, ϕ respectively. From 12.2 (ii) we can assume that $\psi \in \Phi(M(\phi, m))$ and $\phi \in \Phi(M(\psi, n))$. Choose $B \subseteq \bigcup \mathcal{F}$ such that $|B \cap I| = m$ $(I \in \mathcal{F})$ and $C \subseteq \bigcup \mathcal{G}$ such that $|C \cap I| = n$ $(I \in \mathcal{G})$. Let C' be the preimage of C under the canonical embedding of $\bigcup \mathcal{G}'$ into $\bigcup \mathcal{G}$, and B' be the preimage of B under the canonical embedding of $\bigcup \mathcal{F}'$ into $\bigcup \mathcal{F}$. By 12.3 (iii), $|B' \cap I| = m$ $(I \in \mathcal{F}')$ and $|C' \cap I| = n$ $(I \in \mathcal{G}')$. Let $M_0 = (M(\phi, m))(\psi, n)$ be constructed by using B and then C'. Let $M_1 = (M(\psi, n))(\phi, m)$ be constructed by using C and then B'. By definition of one-step shrinking and 12.3 (vi)

$$a \in M_0 \equiv a \in M(\phi, m) \mathbin{\&} \mathrm{cl}^{\mathcal{G}'}(\{a\}) \subseteq C'$$

$$\equiv \mathrm{cl}^{\mathcal{F}}(\{a\}) \subseteq B \mathbin{\&} \mathrm{cl}^{\mathcal{G}}(\{a\}) \subseteq C.$$

By symmetry $M_0 = M_1$. But by the same argument as was used in the proof of 12.4 any two possibilities for M_i $(i < 2)$ are conjugate as subsets of M so we are done.

We recall some definitions from the Introduction. The one-step shrinking which yields $M' = M(\phi, m)$ from M is called *normal* if $\Phi(M') = \Phi(M)$, $\approx_{M'} = \approx_M$, $d_{M'}(\phi) = m$, and $d_{M'}(\psi) = d_M(\psi)$ for all $\psi \in \Phi(M)$ such that $\phi \not\approx_M \psi$. We say that *N is obtained by shrinking M* if N can be obtained from M by a finite sequence of normal one-step shrinkings. We say that N is obtained from M by a *smooth shrinking* if for every $d : \Phi(M) \to \omega$ such that $d_N \leqq d \leqq d_M$ and $d \notin \{d_M, d_N\}$ there exists M' such that $d_{M'} = d$, M' is obtained by shrinking M, and N is obtained by shrinking M'.

LEMMA 12.6. (i) *Any one-step shrinking of $M \in \boldsymbol{H}$ is normal if the target dimension is sufficiently large compared with $l(M)$.*

(ii) *If $N \in \boldsymbol{H}$ is obtained by shrinking $M \in \boldsymbol{H}$, then that shrinking is smooth provided all the target dimensions in the passage from M to N are sufficiently large compared with $l(M)$.*

PROOF. (i) Immediate from 12.2 (ii) and 12.3 (v).

(ii) This is clear from 12.4 and 12.5 because any such shrinking can be viewed as a sequence of normal one-step shrinkings each of which reduces one of the dimensions by exactly one. These one-step shrinkings can be rearranged in any order without the resulting structure changing.

13. The basic stretching lemma

In the last section we discussed a way of shrinking structures. We would like to show that in certain circumstances the shrinking process can be reversed in a unique fashion. More precisely we want to show that if $M \in \boldsymbol{H}$, $\boldsymbol{\phi} \in \Phi(M)$, and $M' = M(\boldsymbol{\phi}, m)$ then the isomorphism type of M over M' is determined by the triple $\langle M', \boldsymbol{\phi}, d_M(\boldsymbol{\phi})\rangle$. Here we shall solve this problem in the simplest case, i.e. where the nice family $\mathcal{F} \in \boldsymbol{F}(M)$ associated with $\boldsymbol{\phi}$ is a singleton.

We introduce some terminology appropriate to this special situation. Let M be an L-structure, where L is a finite relational language but M is not necessarily homogeneous for L. Let $x = \langle x_0, x_1, x_2, x_3\rangle$. A quantifier-free L-formula $\psi(x)$ is called *nice for M* if there exist $\boldsymbol{\phi} = \langle \phi_0, \phi_1\rangle \in \Phi(M)$ and an indiscernible set $I \subseteq M^*$ such that $\psi = \phi_1$ and $\{I\}$ is the nice family associated with $\boldsymbol{\phi}$. The indiscernible set I is called a *nice indiscernible set* attached to M. We say that I is defined by the formula $\psi(x)$. The *index* $i(I)$ *of* I is defined to be $\sup\{|\mathrm{cl}^I(\{a\})| : a \in M\}$ and is assumed to be finite. We also call $i(I)$ the *index of* ψ *in* M and write it $i_M(\psi)$.

The set of nice formulas for M will be denoted $\Psi(M)$ and the set of nice indiscernible sets attached to M will be denoted $\boldsymbol{I}(M)$. The notion of equivalence for nice formulas is analogous to that for nice pairs and is also denoted $\approx_M$.

Previously in the Introduction and §11 and §12 we considered nice pairs and nice families only in the context of homogeneous structures. However the same definitions make sense for any L-structure. If $\psi(x)$ is a nice formula for the L-structure M defining the nice indiscernible set I, then we write $d_M(\psi) = d(I) = |I|$ and call this the ψ*-dimension of* M and simultaneously the *dimension of* I.

Let M be an L-structure, ψ be a nice formula for M defining a nice indiscernible set I, and $m < d_M(\psi)$. By $M(\psi, m)$, and also by $M(I, m)$, we denote the substructure of M whose universe is $\{a \in M : \mathrm{cl}^I(\{a\}) \subseteq B\}$, where $B \subseteq I$ has cardinality m. Since there are many choices for B, $M(\psi, m)$ is not unique if $m > 0$. However, any two possibilities are conjugate in M, because I is strongly indiscernible.

We shall now formulate the analogue of 12.1 for the present context. Recall that a substructure $N \subseteq M$ is homogeneous if it is full and for every $\alpha \in \mathrm{aut}(M)$ and finite $A \subseteq N$ such that $\alpha(A) \subseteq N$ there exists $\beta \in \mathrm{aut}(M)$ such that

$\beta \restriction A = \alpha \restriction A$ and $\beta(N) = N$. Recall that $N \subseteq M$ is a full substructure if for every relation R 0-definable in M of any arity the restriction of R to N is 0-definable in N. In §12 we were assuming that $M \in \boldsymbol{H}$ in which case a substructure is automatically full. The following is obtained by essentially the same argument as 12.1. Recall that $\boldsymbol{P}$ is the class of structures for which the given language and the canonical language are equivalent.

LEMMA 13.1. *Let $M \in \boldsymbol{P}$ be ω-stable, $\psi \in \Psi(M)$, $I \in \boldsymbol{I}(M)$ be defined by ψ, $m < d_M(\psi)$, and $M' = M(\psi, m)$ be a full substructure of M, then M' has the following properties:*

(i) *M' is a homogeneous substructure of M.*
(ii) *M is prime over M'.*
(iii) *Two tuples in M' are conjugate in M' iff they are conjugate in M.*
(iv) *Any automorphism of M' extends to an automorphism of M.*
(v) *If $\bar{a} \in M$, then $\mathrm{tp}(\bar{a})$ is realized in M' iff*

$$\left| \bigcup \{ \mathrm{cl}^I(\{a\}) : a \in \mathrm{rng}(\bar{a}) \} \right| < m.$$

Let $M \in \boldsymbol{P}$, $\psi \in \Psi(M)$, and $I \in \boldsymbol{I}(M)$ be defined by ψ. If $M' \subseteq M$ is a full substructure, $i_{M'}(\psi) = i_M(\psi)$, $M' = M(\psi, m)$ for some $m < d_M(\psi)$, and $I' \in \boldsymbol{I}(M')$ is defined by ψ, then we say that *M' is obtained from M by shrinking I to I'*. Simultaneously we say *M is obtained from M' by stretching I' to I.*

Before passing to the main result of this section we make one observation whose proof is left to the reader. Clause (v) in the definition of nice family is used here.

LEMMA 13.2. *Let $M' \in \boldsymbol{P}$ and I'_0, I'_1 be equivalent nice indiscernible sets attached to M'. If $M \in \boldsymbol{P}$ is obtained from M' by stretching I'_0 to I_0, then there exists I_1 such that M is also obtained from M' by stretching I'_1 to I_1.*

We now come to the fundamental lemma on stretching.

LEMMA 13.3. *Let L be a finite relational language, $M' \in \boldsymbol{H}(L)$, $\psi \in \Psi(M')$, and*

$$i_{M'}(\psi) \cdot \max\{a(M'), 8\} \leqq d_{M'}(\psi)/3 < \omega.$$

For every k, $d_{M'}(\psi) < k \leqq \omega$, there exists an ω-stable structure $M \in \boldsymbol{C}(L)$ such that $\psi \in \Psi(M)$, $i_M(\psi) = i_{M'}(\psi)$, $d_M(\psi) = k$, $M' = M(\psi, d_{M'}(\psi))$, and M' is a full substructure of M. Further, the isomorphism type of M over M' is uniquely determined by k.

PROOF. Let $I' \in \boldsymbol{I}(M')$ be defined by ψ. Let $I'_0 \subseteq I'_1 \subseteq I'_2 \subseteq I'$ be chosen such that $|I'_0| = mn$, $|I'_1| = (m+1)n$, and $|I'_2| = 2mn$, where m denotes $\max\{a(M'), 8\}$, and n denotes $i_{M'}(\psi)$. Denote mn by l. Let $I'_1 - I'_0 = \{b'_i : i < n\}$ and $I'_2 - I'_0 = \{b'_i : i < l\}$. For $i < 3$ let $M'_i \subseteq M'$ be the homogeneous substructure defined by

$$M'_i = \{a \in M' : \mathrm{cl}^{I'}(\{a\}) \subseteq I'_i\}.$$

Let $I^*_i \subseteq (M'_i)^*$ be defined by ψ, i.e. $I^*_i = (M'_i)^2/E_i$ where $E_i \in \mathscr{E}((M'_i)^2)$ is defined by ψ. There is a canonical embedding of I^*_i into I' which is onto I'_i. It is easy to see that $I^*_i \in \boldsymbol{I}(M'_i)$. For $i < j < 3$ there is a canonical embedding of I^*_i into I^*_j. Let the range of this canonical embedding be denoted $I'_{i,j}$.

The idea of the proof is as follows. We find a way of reconstructing the permutation group $\mathrm{aut}(M'_2, \{I'_{0,2}\})$ from $\mathrm{aut}(M'_1, \{I'_{0,1}\})$. This gives a natural way of stretching the nice indiscernible set $I^*_2 - I'_{0,2}$ attached to $(M'_2, \{I'_{0,2}\})$. Let N_k denote the structure obtained from $(M'_2, \{I'_{0,2}\})$ by stretching $I^*_2 - I'_{0,2}$ to a set of power k, $l < k \leqq \omega$. For this stretching $(M'_2, \{I'_{0,2}\})$ is given its canonical language which is also the language of N_k although N_k may not be homogeneous for it as far as we know. Every m-tuple in N_k is conjugate in N_k to an m-tuple of M'_2. Hence there is a unique quantifier-free way to define $L = L(M')$ in $L(N_k)$ so that M'_2 is a reduct of a substructure of N_k.

We show that if ω-stable $M \in \boldsymbol{P}(L)$ is obtained from M'_2 by stretching I^*_2 to a set of power $l + k$, then there is an isomorphism between M and $N_k \mid L$ which is the identity on M'_2. This basic uniqueness result is Claim 10.

Let N_l denote $(M'_2, \{I'_{0,2}\})$. In $L(N_l)$ there is a quantifier-free formula $\theta(x_0, x_1)$ such that $N_l \models \theta(a_0, a_1)$ if $\langle a_0, a_1 \rangle \in (M'_2)^2$ represents a member of $I'_{0,2}$. Then the formula $\chi = \psi \wedge \theta(x_0, x_1) \wedge \theta(x_2, x_3)$ is nice for N_l and defines $I'_{0,2}$ as a nice indiscernible set attached to N_l. Let $J_k \subseteq N^*_k$ be defined by χ. Then the canonical embedding of $I'_{0,2}$ into J_k is a bijection. Let $I_k \subseteq N^*_k$ be the set defined by ψ which is attached to N_k, i.e. $I_k = (N_k)^2/E$ where $E \in \mathscr{E}((N_k)^2)$ is defined by ψ.

From the way N_k is built it is clear that $N_l, N_{l+1}, \dots$ can be chosen so that they form an ascending chain. Let $M_k = N_k \restriction L$ $(k > l)$. Then $\langle M_l : l \leqq k < \omega \rangle$ is an ascending chain of L-structures.

The next step is to show that if $l \leqq j \leqq k$, $\alpha \in \mathrm{aut}(M_j)$ and $\alpha(J_j) = J_j$, then there exists $\beta \in \mathrm{aut}(N_k)$ such that $\beta \supseteq \alpha$. From this we deduce that $M_k \in \boldsymbol{C}$ and M_k is ω-stable. The same is true for N_k.

We also show that I_k is a nice indiscernible set attached to M_k and that $i(I_k) = n$. To this end we observe that J_k is indiscernible in N^*_k and $\mathrm{cl}^{J_k}(\{a\})$ exists for all $a \in N_k$. Let M^+_k be the reduct to L of the substructure of N_{k+l} with universe

$$\{a \in N_{k+l} : \mathrm{cl}^{J_{k+l}}(\{a\}) = \varnothing\}.$$

Then M_k and M_k^+ are isomorphic by Claim 10. The desired conclusion about I_k follows easily.

For $l \leqq j < k$ let $I_{j,k}$ denote the range of the canonical embedding of I_j into I_k. Then

$$M_j = \{a \in M_k : \mathrm{cl}_{M_k}^{I_k}(\{a\}) \subseteq I_{j,k}\}.$$

Hence for any $\alpha \in \mathrm{aut}(M_k)$ and finite $A \subseteq M_j$ such that $\alpha(A) \subseteq M_j$ there exists $\beta \in \mathrm{aut}(M_k)$ such that $\beta \restriction A = \alpha \restriction A$ and $\beta(M_j) = M_j$. We also show that every $\alpha \in \mathrm{aut}(M_l)$ extends to $\beta \in \mathrm{aut}(M_k)$. We conclude that $\langle M_k : l \leqq k < \omega \rangle$ is a homogeneous chain and that automorphisms extend upwards.

The conclusions of the lemma follow easily. The given M' can be identified with $M_{|I'|-l}$ and then M_{k-l} will do for M. Let N be another possibility for M, then Claim 10 yields an isomorphism $\alpha : N \to M$ with $M'_2 = M_l \subseteq \mathrm{fix}(\alpha)$. There exists $\beta \in \mathrm{aut}(M)$ such that $M' \subseteq \mathrm{fix}(\beta\alpha)$ and so we finish.

We now begin to carry out the proof plan just sketched.

Claim 1. *For $i < 3$ every $\alpha \in \mathrm{aut}(M'_i)$ has an extension in* $\mathrm{aut}(M')$, *and for $i < j < 3$ every $\alpha \in \mathrm{aut}(M'_i)$ has an extension in* $\mathrm{aut}(M'_j)$.

Proof. Immediate from 12.1 (iv) since $M' \in \boldsymbol{H}(L)$ and the M'_i are obtained by shrinking.

For $w \subseteq l$ with $|w| \leqq n$ define

$$A(w) = \{a \in M' : \{b'_i : i \in w\} \subseteq \mathrm{cl}^{I'}(\{a\}) \subseteq I'_0 \cup \{b'_i : i \in w\}\}.$$

Then $\{A(w) : w \subseteq l, |w| \leqq n\}$ is a partition of M'_2 and $A(\emptyset) = M'_0$. Let

$$A_0(w) = \bigcup \{A(v) : v \subsetneqq w\},$$

$$A_1(w) = \bigcup \{A(v) : v \subseteq l, |v| \leqq |w|, \text{ and } v \neq w\}.$$

The key to the whole proof is

Claim 2. *If $w \subseteq l$ and $|w| \leqq n$, then*

$$\mathrm{aut}_{M'_2}(A(w), A_0(w)) = \mathrm{aut}_{M'_2}(A(w), A_1(w)).$$

Proof. Recall that for $A, B \subseteq M$ by $\mathrm{aut}_M(A, B)$ we denote the set of all $\pi \in \mathrm{perm}(A)$ such that for some $\alpha \in \mathrm{aut}(M)$, $\pi \subseteq \alpha$ and $B \subseteq \mathrm{fix}(\alpha)$. If $w = \emptyset$ the conclusion is immediate, so suppose $w \neq \emptyset$.

Consider $\alpha \in \mathrm{aut}(M')$ such that $\alpha(M'_2) = M'_2$, $\alpha(A(w)) = A(w)$, and $A_0(w) \subseteq \mathrm{fix}(\alpha)$. Every member of I'_0 has a representative in $A_0(w)^2$. Hence $I'_0 \subseteq \mathrm{fix}(\alpha)$. If

$|w|>1$ every member of $\{b'_j : j \in w\}$ has a representative in $A_0(w)^2$, whence $\{b'_j : j \in w\} \subseteq \mathrm{fix}(\alpha)$. If $|w| = 1$ then $w = \{j\}$ for some $j < l$ and $b'_j \in \mathrm{fix}(\alpha)$ since $\alpha(A(w)) = A(w)$. Thus in any case $I'_0 \cup \{b'_j : j \in w\} \subseteq \mathrm{fix}(\alpha)$.

Let $a_0, \ldots, a_{i-1} \in A_1(w)$ where $i < m$ and let $a_i, \ldots, a_{m-1} \in A(w)$. For $i \leqq h < m$

$$\{b'_j : j \in w\} \subseteq \mathrm{cl}^{I'}(\{a_h\}) \subseteq I'_0 \cup \{b'_j : j \in w\},$$

and so $\mathrm{cl}^{I'}(\{\alpha(a_h)\}) = \mathrm{cl}^{I'}(\{a_h\})$ for $i \leqq h < m$. Let

$$B_0 = \bigcup\{\mathrm{cl}^{I'}(\{a_j\}) : i \leqq j < m\} = \bigcup\{\mathrm{cl}^{I'}(\{\alpha(a_j)\}) : i \leqq j < m\}.$$

Then $B_0 \subseteq I'_0 \cup \{b'_j : j \in w\}$ and $|B_0| \leqq (m-i)n$. Let $B_1 = \bigcup\{\mathrm{cl}^{I'}(\{a_j\}) : j < i\}$. Then $B_1 \subseteq I'_0 \cup \{b'_j : j < l\}$ and $|B_1| \leqq in$. There exists $\pi \in \mathrm{perm}(I')$ such that $\pi(I'_2) = I'_2$, $B_0 \subseteq \mathrm{fix}(\pi)$, and $\pi(B_1 - \{b'_j : j \in w\}) \subseteq I'_0$. Since I' is a nice indiscernible set, by 11.2 π can be extended to $\gamma \in \mathrm{aut}(M')$ such that $a_j, \alpha(a_j) \in \mathrm{fix}(\gamma)$ for $i \leqq j < m$. Since $\gamma(I'_2) = I'_2$, $\gamma \restriction M'_2 \in \mathrm{aut}(M'_2)$. Since $\gamma(B_1 - \{b'_j : j \in w\}) \subseteq I'$ and $\{b'_j : j \in w\} \subseteq \mathrm{fix}(\gamma)$, $\gamma(a_j) \in A_0(w)$ for $j < i$. But $A_0(w) \subseteq \mathrm{fix}(\alpha)$ and so

$$\mathrm{tp}(\gamma(a_0), \ldots, \gamma(a_{i-1}), a_i, \ldots, a_{m-1}) = \mathrm{tp}(\gamma(a_0), \ldots, \gamma(a_{i-1}), \alpha(a_i), \ldots, \alpha(a_{m-1})).$$

Applying γ^{-1} we obtain

$$\mathrm{tp}(a_0, \ldots, a_{i-1}, a_i, \ldots, a_{m-1}) = \mathrm{tp}(a_0, \ldots, a_{i-1}, \alpha(a_i), \ldots, \alpha(a_{m-1})).$$

But $a_0, \ldots, a_{i-1}$ are arbitrary in $A_1(w)$ as are $a_i, \ldots, a_{m-1}$ in $A(w)$, and $a(M') \leqq m$. Therefore for any tuple $\bar{c} \in A(w)$, $\mathrm{tp}(\bar{c} \mid A_1(w)) = \mathrm{tp}(\alpha(\bar{c}) \mid A_1(w))$. Now $A(w) \cup A_1(w)$ is 0-definable in $((M')^*, I'_2)$ and the map π fixing each element of $A_1(w)$ and agreeing with α on $A(w)$ is elementary with respect to this model. By 3.4 π extends to $\beta \in \mathrm{aut}(M', I'_2)$. Since $\beta \restriction M'_2 \in \mathrm{aut}(M'_2)$ we are done.

Claim 2 enables us to show that in a sense the permutation group $\mathrm{aut}(M'_2, \{I'_{0,2}\})$ is coded in $\mathrm{aut}(M'_1, \{I'_{0,1}\})$.

If $w \subseteq l$ and $\sigma : w \to l$ we write $\sigma \uparrow$ to mean that σ is strictly increasing. We shall construct a family

$$\alpha(w, \sigma(w)) : A(w) \to A(\sigma(w)) \qquad (w \subseteq l, |w| \leqq n, \sigma : w \to l, \text{and } \sigma \uparrow)$$

of bijections such that for all such w and σ

$$\alpha(w, \sigma(w))(b'_i) = b'_{\sigma(i)} \qquad (i \in w),$$

$$\alpha(w, \sigma(w)) = \alpha(\sigma(w), |w|)^{-1} \alpha(w, |w|),$$

and the map

$$\beta(w,\sigma(w)) = \bigcup\{\alpha(u,\sigma(u)) : u \subseteq w\}$$

can be extended to an automorphism of M_2'.

The construction proceeds by induction on $|w|$. Let $\alpha(\emptyset,\emptyset)$ be the identity on $A(\emptyset)$. Now fix $w \subseteq l$ and $\sigma : w \to l$ such that $|w| \leqq n$, $w \neq \emptyset$, and $\sigma \uparrow$. For induction suppose that $\alpha(u,\tau(u))$ has been defined for all $\langle u,\tau\rangle$ with $u \subseteq l$, $|u| < |w|$, $\tau : w \to l$, and $\tau \uparrow$, such that $\alpha(u,\tau(u)) = \alpha(\tau(u),|u|)^{-1}\alpha(u,|u|)$ and for all $\langle v,\pi\rangle$ with $v \subseteq l$, $\pi : v \to l$, and $\pi \uparrow$, the map

$$\bigcup\{\alpha(u,\pi(u)) : u \subseteq v \text{ and } |u| < |w|\}$$

can be extended to an automorphism of M_2'.

If $w = |w|$ let $\alpha(w,|w|)$ be the identity. Otherwise let $\pi : w \to |w|$ be the unique map such that $\pi \uparrow$ and let $\gamma \in \operatorname{aut}(M_2')$ extend $\alpha(u,\pi(u))$ for all $u \subsetneqq w$. Let $\gamma' \in \operatorname{aut}(M')$ extend γ. Suppose $|w| > 1$ and $i \in w$, then $\alpha(\emptyset,\emptyset)$, $\alpha(\{i\},\{\pi(i)\}) \subseteq \gamma$, whence $\gamma(A(\{i\})) = A(\{\pi(i)\})$ and $\gamma(A(\emptyset)) = A(\emptyset)$. Thus in this case $\gamma'(b_i') = b_{\pi(i)}'$ for all $i \in w$. Now suppose $|w| = 1$. Since I' is a nice indiscernible set there exists $\gamma'' \in \operatorname{aut}(M')$ such that $\gamma''(b_i') = b_{\pi(i)}'$ $(i \in w)$, $\gamma''(I_2') = I_2'$, and $A(\emptyset) \subseteq \operatorname{fix}(\gamma'')$. We have $\gamma'' \restriction M_2' \in \operatorname{aut}(M_2')$ and so $\gamma'' \restriction M_2'$ is a possible choice for γ and γ'' is a possible choice for γ'. Hence whatever $|w|$ is we can assume $\gamma'(b_i') = b_{\pi(i)}'$ $(i \in w)$. It follows that $\gamma(A(w)) = A(|w|)$.

Define $\alpha(w,|w|) = \gamma \restriction A(w)$ and generally let $\alpha(w,\sigma(w)) = \alpha(\sigma(w),|w|)^{-1}\alpha(w,|w|)$. From the definition it is clear that $\beta(w,|w|) \subseteq \gamma$ can be extended to an automorphism of M_2'. Hence $\beta(w,\sigma(w)) = \beta(\sigma(w),|w|)^{-1}\beta(w,|w|)$ can also be extended to an automorphism of M_2'.

To complete the induction step fix $v \subseteq l$ and $\pi : v \to l$ such that $\pi \uparrow$. We must show that

$$\bigcup\{\alpha(u,\pi(u)) : u \subseteq v \text{ and } |u| \leqq |w|\}$$

can be extended to an automorphism of M_2'. Let $\delta \in \operatorname{aut}(M_2')$ extend $\alpha(u,\pi(u))$ for all $u \subseteq v$ with $|u| < |w|$, and $\delta' \in \operatorname{aut}(M')$ extend δ. Suppose $|v| \geqq |w|$, otherwise there is nothing to prove. If $|w| > 1$ we automatically have $\delta'(b_i') = b_{\pi(i)}'$ $(i \in v)$. If $|w| = 1$ we can choose δ and δ' so that the same is true. In any case we obtain $\delta(A(u)) = A(\pi(u))$ for all $u \subseteq v$. For $u \subseteq v$ with $|u| = |w|$ let $\xi(u,\pi(u)) \in \operatorname{aut}(M_2')$ extend $\beta(u,\pi(u))$. Notice that $A_0(u) \subseteq \operatorname{fix}(\delta^{-1}\xi(u,\pi(u)))$. From Claim 2 there exists $\eta \in \operatorname{aut}(M_2')$ such that $A_1(u) \subseteq \operatorname{fix}(\eta)$ and η agrees with $\delta^{-1}\xi(u,\pi(u))$ on $A(u)$. Then $\delta\eta$ agrees with δ on $A_1(u)$ and hence on $A(t)$ for all $t \subseteq l$ such that $|t| \leqq |w|$ and $t \neq u$. Also $\delta\eta$ agrees with $\xi(u,\pi(u))$ and hence with $\alpha(u, \pi(u))$ on $A(u)$. Repeating this process once for each $u \subseteq v$, $|u| = |w|$, we can find $\delta^* \in \text{aut}(M_2')$ such that $\delta^* \restriction A(u) = \delta \restriction A(u)$ for all $u \subseteq v$ with $|u| < |w|$, and $\delta^* \restriction A(u) = \alpha(u, \pi(u)) \restriction A(u)$ for all $u \subseteq v$ with $|u| = |w|$. Since δ^* is just the kind of automorphism required the induction step is complete and so is the definition of the maps $\alpha(w, \sigma(w))$.

The amount of freedom in the definition of the maps $\alpha(w, \sigma(w))$ is inessential in the following sense.

CLAIM 3. *Let $\alpha(w, \sigma(w))$ and $\alpha'(w, \sigma(w))$ ($w \subseteq l$, $|w| \leqq n$, $\sigma : w \to l$, and $\sigma \uparrow$) be two possible families of bijections satisfying the requirements laid down above. There exists $\gamma \in \text{aut}(M_2', M_0' \cup I_2^*)$ such that*

$$\alpha'(w, \sigma(w)) = (\gamma^{-1}\alpha(w, \sigma(w))\gamma) \restriction A(w)$$

for all $w \subseteq l$ and $\sigma : w \to l$ with $|w| \leqq n$ and $\sigma \uparrow$.

PROOF. Since $\alpha(w, \sigma(w)) = \alpha(\sigma(w), |w|)^{-1}\alpha(w, |w|)$ and similarly for α' it is sufficient to find $\gamma \in \text{aut}(M_2', M_0' \cup I_2^*)$ such that $\alpha'(w, |w|) = (\alpha(w, |w|)\gamma) \restriction A(w)$ for all $w \subseteq l$ with $|w| \leqq n$. Further taking $w = \emptyset$ it is clear that $\alpha(\emptyset, \emptyset)$ and $\alpha'(\emptyset, \emptyset)$ are both the identity on M_0'. Define $\gamma \in \text{perm}(M_2')$ by

$$\gamma \restriction A(w) = \alpha(w, |w|)^{-1}\alpha'(w, |w|) \qquad (w \subseteq l, |w| \leqq n).$$

Then $\gamma \restriction M_0'$ is the identity. Let

$$A[j] = \bigcup\{A(w) : w \subseteq l, |w| \leqq j\} \qquad (j \leqq n).$$

By induction on j we show that $\gamma \restriction A[j]$ can be extended to an automorphism of M_2' for all $j \leqq n$. This induction is similar to the previous one so we omit the details. Since $A[n] = M_2'$ we have $\gamma \in \text{aut}(M_2')$. Since $\gamma(A(w)) = A(w)$ for all $w \subseteq l$ with $|w| \leqq n$, $\gamma \restriction I_2^*$ is the identity. This completes the proof of the claim.

We are still working towards the reconstruction of $\text{aut}(M_2', \{I_{0,2}'\})$ from $\text{aut}(M_1', \{I_{0,1}'\})$. For $i < n$ let $b_{i,1}$ denote the preimage of b_i' under the canonical embedding of I_1^* into I'. For $i < l$ let $b_{i,2}$ denote the preimage of b_i' under the canonical embedding of I_2^* into I'. The next step is to define a family γ_π ($\pi \in \text{perm}(n)$) of automorphisms of M_1' such that $A(\emptyset) \subseteq \text{fix}(\gamma_\pi)$ and $\gamma_\pi(b_{i,1}) = b_{\pi(i),1}$ for all $i < n$. The definition of $\gamma_\pi \restriction A(w)$, $w \subseteq n$, proceeds by induction on $|w|$ simultaneously for all π. For $|w| = 0$ let $\gamma_\pi \restriction A(\emptyset)$ be the identity and for $w = \{i\}$ where $i < n$ let $\gamma_\pi \restriction A(\{i\})$ be $\alpha(\{i\}, \{\pi(i)\})$. For induction suppose $1 < j \leqq n$, γ_π has been defined on $A[j-1] \cap M_1'$ for all π, and $\gamma_\pi \restriction (A[j-1] \cap M_1')$ has an extension in $\text{aut}(M_1')$. For $\pi \in \text{perm}(n)$ let $\delta \in$

$\operatorname{aut}(M_1')$ extend $\gamma_\pi \restriction (A[j-1] \cap M_1')$. Then we define $\gamma_\pi \restriction A(j) = \delta \restriction A(j)$. For $\pi \in \operatorname{perm}(n)$ and arbitrary $w \subseteq n$ with $|w| = j$ let $\sigma : w \to j$, $\tau : \pi(w) \to j$ be the unique strictly increasing bijections, and let $\rho \in \operatorname{perm}(n)$ be defined by

$$\rho \restriction j = (\tau\pi\sigma^{-1}) \restriction j \ \& \ \{i : j \leqq i < n\} \subseteq \operatorname{fix}(\rho).$$

Notice the way in which ρ depends on w and π since this will be important below. Define

$$(\#) \qquad \gamma_\pi \restriction A(w) = (\alpha(j, \pi(w))\gamma_\rho \alpha(w, j)) \restriction A(w).$$

Observe that $(\#)$ holds when $j = 0, 1$ and so for induction we can assume that $(\#)$ holds when j is replaced by any $k < j$, w by any $v \subseteq n$ with $|v| = k$, and ρ by the appropriate permutation found from v and π.

To see that this definition of $\gamma_\pi \restriction A(w)$ $(w \subseteq n, |w| = j)$ is appropriate we first verify that for $u \subsetneqq w$

$$(*) \qquad \gamma_\pi \restriction A(u) = (\beta(j, \pi(w))\gamma_\rho \beta(w, j)) \restriction A(u).$$

(Note that $(\#)$ says $(*)$ holds for $u = w$.) Let $|u| = h$ and $\xi \in \operatorname{perm}(n)$ be defined from u and π as ρ was from w and π. By the induction hypothesis

$$\gamma_\pi \restriction A(u) = (\alpha(h, \pi(u))\gamma_\xi \alpha(u, h)) \restriction A(u).$$

Let $v = \sigma(u)$. The permutation defined from v and ρ as ρ was from w and π is again ξ. (We leave the reader to check this for himself.) Again by the induction hypothesis

$$\gamma_\rho \restriction A(v) = (\alpha(h, \rho(v))\gamma_\xi \alpha(v, h)) \restriction A(v).$$

Consider the right-hand side of $(*)$. By definition of $\beta(w, j)$, $\beta(w, j) \restriction A(u) = \alpha(u, v)$. Next γ_ρ maps $A(v)$ onto $A(\rho(v))$, and $\beta(j, \pi(w)) \restriction A(\rho(v)) = \alpha(\rho(v), \pi(u))$ by definition of $\beta(j, \pi(w))$ remembering that $\rho \restriction j = (\tau\pi\sigma^{-1}) \restriction j$. Thus the right-hand side of $(*)$ can be written

$$\alpha(\rho(v), \pi(u))\gamma_\rho \alpha(u, v).$$

When we substitute for γ_ρ this becomes

$$\alpha(\rho(v), \pi(u))\alpha(h, \rho(v))\gamma_\xi \alpha(v, h)\alpha(u, v).$$

But $\alpha(w, y)\alpha(x, w) = \alpha(x, y)$ for all admissible arguments x, y, w, and so the last expression is equal to that for $\gamma_\pi \restriction A(u)$ given by the induction hypothesis. This completes the verification of $(*)$.

From $(*)$ $\gamma_\pi \restriction (A(w) \cup A_0(w))$ can be extended to an automorphism of M'

because this is true of $\beta(j,\pi(w))$, γ_ρ, and $\beta(w,j)$. Claim 2 allows us to deduce that $\gamma_\pi \restriction (A[j]\cap M_1')$ can be extended to $\alpha \in \operatorname{aut}(M')$. But $\gamma_\pi(A(\{i\}))=A(\{\pi(i)\})$ and so $\alpha(b_i')=b_{\pi(i)}'$ for $i<n$. Hence $\alpha(M_1')=M_1'$ which shows that $\gamma_\pi\restriction(A[j]\cap M_1')$ can be extended to an automorphism of M_1'. This completes the induction step and confirms that the desired family γ_π $(\pi\in\operatorname{perm}(n))$ of automorphisms of M_1' can be found.

CLAIM 4. *There is a unique family γ_π $(\pi\in\operatorname{perm}(l))$ of automorphisms of M_2' such that*

$$(\#)\qquad \gamma_\pi\restriction A(w)=(\alpha(j,\pi(w))\gamma_\rho\alpha(w,j))\restriction A(w)$$

for all $\pi\in\operatorname{perm}(l)$ and $w\subseteq l$ with $|w|=j\leqq n$, where $\rho\in\operatorname{perm}(n)$ is defined from w and π as above.

PROOF. Since $M_2'=\bigcup\{A(w): w\subseteq l,\ |w|\leqq n\}$ the relation $(\#)$ defines the maps γ_π $(\pi\in\operatorname{perm}(l))$ explicitly. To see that $\gamma_\pi\in\operatorname{aut}(M_2')$ we repeat virtually the same argument we used to show that $\gamma_\sigma\in\operatorname{aut}(M_1')$ for $\sigma\in\operatorname{perm}(n)$.

CLAIM 5. *Let $\alpha\in\operatorname{perm}(M_2')$ and $\alpha(A(w))=A(w)$ for all $w\subseteq l$, with $|w|\leqq n$. Then $\alpha\in\operatorname{aut}(M_2')$ if and only if the map $\beta(w,|w|)\alpha\beta(|w|,w)$ has an extension in* $\operatorname{aut}(M_1')$ *for all $w\subseteq l$ with $|w|\leqq n$.*

PROOF. Let $\alpha\in\operatorname{aut}(M_2')$ satisfy the hypothesis. Then $\beta(w,|w|)\alpha\beta(|w|,w)$ maps $A(|w|)\cup A_0(|w|)$ onto itself and can be extended to $\gamma\in\operatorname{aut}(M_2')$ since each of the three terms in the product can. By Claim 1, γ extends to $\gamma'\in\operatorname{aut}(M')$. For all $u\subseteq|w|$, $\beta(w,|w|)\alpha\beta(|w|,w)$ maps $A(u)$ onto $A(u)$, whence $\gamma'(b_i')=b_i'$ for $i\leqq|w|$ and $\gamma(I_0')=I_0'$. Since I' is nice there exists $\delta\in\operatorname{aut}(M')$ such that $\delta\gamma'\restriction I'$ is the identity and $A(|w|)\cup A_0(|w|)\subseteq\operatorname{fix}(\delta)$. Now $\delta\gamma'\restriction M_1'\in\operatorname{aut}(M_1')$ and $\delta\gamma'\supseteq\beta(w,|w|)\alpha\beta(|w|,w)$ so that "only if" part is done.

For the rest let $\alpha\in\operatorname{perm}(M_2')$ satisfy: $\alpha(A(w))=A(w)$ and $\beta(w,|w|)\alpha\beta(|w|,w)$ has an extension in $\operatorname{aut}(M_1')$ for all $w\subseteq l$ with $|w|\leqq n$. Any automorphism of M_1 can be extended to an automorphism of M_2'. Therefore $\beta(w,|w|)\alpha\beta(|w|,w)$ has an extension $\gamma\in\operatorname{aut}(M_2')$. Since

$$\alpha\restriction(A(w)\cup A_0(w))=\beta(|w|,w)\gamma\beta(w,|w|),$$

$\alpha\restriction(A(w)\cup A_0(w))$ has an extension in $\operatorname{aut}(M_2')$. Using Claim 2 we see by induction on j that $\alpha\restriction A[j]$ has an extension in $\operatorname{aut}(M_2')$ for all $j\leqq n$. Taking $j=n$ we are done.

We conclude that given the mappings $\alpha(w,|w|)$ ($w \subseteq l$, $|w| \leqq n$) and the permutation group aut($M'_1, \{I'_{0,1}\}$) we can reconstruct aut($M'_2, \{I'_{0,2}\}$), because Claim 4 gives us the family of automorphisms γ_π ($\pi \in$ perm(l)) of M'_2 and Claim 5 gives aut($M'_2, \{I'_{0,2}\} \cup (I^*_2 - I'_{0,2})$). Of course, the family of automorphisms γ_π ($\pi \in$ perm(n)) of M'_1 is not uniquely determined, but any choice of this family serves our purpose.

We begin a new phase of the proof. ($M'_2, \{I'_{0,2}\}$) is obtained from ($M'_1, \{I'_{0,1}\}$) by stretching $I^*_1 - I'_{0,1}$ to $I^*_2 - I'_{0,2}$, and we have found a way to construct ($M'_2, \{I'_{0,2}\}$) given ($M'_1, \{I'_{0,1}\}$). We now use this method to stretch $I^*_1 - I'_{0,1}$ to an arbitrarily large set.

Let ($M'_2, \{I'_{0,2}\}$) be denoted N_l. Let the automorphisms γ_π ($\pi \in$ perm(n)) be given and also the maps $\alpha(w,|w|)$ ($w \subseteq l$, $|w| \leqq n$). For each $k > l$ we construct a homogeneous extension N_k of N_l as follows. For all $w \subseteq k$ with $|w| \leqq n$ and $w \not\subseteq l$ choose pairwise disjoint sets $A(w)$ all disjoint from N_l such that $|A(w)| = |A(|w|)|$. For each such w fix arbitrarily a bijection $\alpha(w,|w|)$: $A(w) \to A(|w|)$ and let $\alpha(|w|, w) = \alpha(w,|w|)^{-1}$. Now $\alpha(w,|w|)$ and $\alpha(|w|, w)$ have been defined for all $w \subseteq k$ with $|w| \leqq n$. Define $\beta(w,|w|)$ and $\beta(|w|, w)$ as before. Let N_k have universe $\bigcup\{A(w): w \subseteq k, |w| \leqq n\}$. For each $\pi \in$ perm(k) define $\gamma_\pi \in$ perm(N_k) by

$$\gamma_\pi \restriction A(w) = (\alpha(|w|, \pi(w))\gamma_\rho\alpha(w,|w|)) \restriction A(w)$$

where $\rho \in$ perm(n) is defined from w and π as before. We can suppose that $\gamma_\pi \restriction A(w)$ is the identity whenever $\pi \restriction w$ is the identity, because γ_ι can be taken to be the identity when ι is the identity on n. Let $G \subseteq$ perm(N_k) consist of all $\alpha \in$ perm(N_k) such that $\alpha(A(w)) = A(w)$ and $\beta(w,|w|)\alpha\beta(|w|, w)$ has an extension in aut(M'_1) for all $w \subseteq k$ with $|w| \leqq n$. Then G is clearly a group. Define N_k as a structure by letting aut(N_k) be the subgroup of perm(N_k) generated by G and the γ_π ($\pi \in$ perm(k)).

Claim 6. $G \lhd$ aut(N_k).

Proof. The key is the identity

$$(*) \qquad \gamma_\pi \restriction (A(w) \cup A_0(w)) = \beta(|w|, \pi(w))\gamma_\rho\beta(w,|w|)$$

which holds for all $\pi \in$ perm(k) and $w \subseteq k$ with $|w| \leqq n$. Here $\rho \in$ perm(n) is defined exactly as before. That is, $\sigma: w \to |w|$ and $\tau: \pi(w) \to |w|$ are the unique strictly increasing functions, $\rho \restriction |w| = (\tau\pi\sigma^{-1}) \restriction |w|$, and $\rho(i) = i$ if $|w| \leqq i < n$. The proof of (*) is the same as before when we were constructing the γ_π ($\pi \in$ perm(n)).

Let $\pi \in \mathrm{perm}(k)$ and $\delta \in G$. It is sufficient to show that $\gamma_\pi^{-1}\delta\gamma_\pi \in G$. Let $w \subseteq k$ and $|w| \leqq n$. By definition of G it is enough to show that

$$\xi = \beta(w, |w|)\gamma_\pi^{-1}\delta\gamma_\pi\beta(|w|, w)$$

has an extension in $\mathrm{aut}(M'_1)$. Substituting for γ_π by (*) and cancelling on the outside we obtain

$$\xi = \gamma_\rho^{-1}\beta(\pi(w), |w|)\delta\beta(|w|, \pi(w))(\gamma_\rho \restriction A(|w|) \cup A_0(|w|)).$$

Now $\beta(\pi(w), |w|)\delta\beta(|w|, \pi(w))$ has an extension in $\mathrm{aut}(M'_1)$ since $\delta \in G$. Therefore ξ has an extension in $\mathrm{aut}(M'_1)$ and we are done.

Claim 7. *If $\sigma, \pi \in \mathrm{perm}(k)$ then $(\gamma_{\sigma\pi})^{-1}\gamma_\sigma\gamma_\pi \in G$.*

Proof. Let $w \subseteq k$ and $|w| = j \leqq n$. From (*) there are automorphisms η, ξ, ζ in $\{\gamma_\rho : \rho \in \mathrm{perm}(n)\}$ such that

$$\gamma_\pi \restriction (A(w) \cup A_0(w)) = \beta(j, \pi(w))\eta\beta(w, j),$$

$$\gamma_\sigma \restriction (A(\pi(w)) \cup A_0(\pi(w))) = \beta(j, \sigma\pi(w))\xi\beta(\pi(w), j),$$

$$\gamma_{\sigma\pi} \restriction (A(w) \cup A_0(w)) = \beta(j, \sigma\pi(w))\zeta\beta(w, j).$$

Hence

$$(\gamma_{\sigma\pi})^{-1}\gamma_\sigma\gamma_\pi \restriction (A(w) \cup A_0(w)) = \beta(j, w)\zeta^{-1}\xi\eta\beta(w, j).$$

It follows that

$$\beta(w, j)(\gamma_{\sigma\pi})^{-1}\gamma_\sigma\gamma_\pi\beta(j, w) \subseteq \zeta^{-1}\xi\eta \in \mathrm{aut}(M'_1).$$

By definition of G this is enough.

From Claim 7, γ_π ($\pi \in \mathrm{perm}(k)$) are a system of representatives for the cosets of G in $\mathrm{aut}(N_k)$ and $(\gamma_\pi/G) \mapsto \pi$ ($\pi \in \mathrm{perm}(k)$) is an isomorphism of $\mathrm{aut}(N_k)/G$ onto $\mathrm{perm}(k)$.

Claim 8. *Every $\alpha \in \mathrm{aut}(N_l)$ has an extension $\beta \in \mathrm{aut}(N_k)$.*

Proof. Let $\alpha \in \mathrm{aut}(N_l)$. Since $N_l = (M'_2, \{I'_{0,2}\})$, α induces a permutation of I_2^* and $\alpha(I'_{0,2}) = I'_{0,2}$. Let $\pi \in \mathrm{perm}(l)$ be the unique permutation such that $\alpha(b_{\pi(i),2}) = b_{i,2}$ $(i < l)$. Then $\gamma_\pi\alpha \in \mathrm{aut}(N_l)$ and $\gamma_\pi\alpha(A(w)) = A(w)$ for all $w \subseteq l$ with $|w| \leqq n$. Let $\sigma \in \mathrm{perm}(k)$ extend π. Then $\gamma_\sigma \supseteq \gamma_\pi$ by inspection of the definitions. If $\gamma_\pi\alpha \subseteq \beta \in \mathrm{aut}(N_k)$ then $\alpha \subseteq (\gamma_\sigma)^{-1}\beta \in \mathrm{aut}(N_k)$. Thus we can assume that $\alpha(A(w)) = A(w)$ for all $w \subseteq l$ with $|w| \leqq n$.

For $j \leqq n$ and $u \subseteq k$ with $|u| \leqq n$ define

$$A(u,j) = \bigcup\{A(w): w \subseteq u \ \&\ |w| \leqq j\}$$

and for any $\sigma: u \to k$ with $\sigma \uparrow$

$$\beta_j(u, \sigma(u)) = \beta(u, \sigma(u)) \restriction A(u,j).$$

By induction on j for $j \leqq n$ we define $\gamma_j \in \mathrm{perm}(A(k,j))$ such that

(i) $\gamma_0 = \alpha \restriction A(\emptyset)$ and $\gamma_j \supseteq \gamma_{j-1}$ $(j > 0)$,

(ii) $\gamma_j(A(w)) = A(w)$ for all $w \subseteq k$ with $|w| \leqq j$,

(iii) $\gamma_j \restriction A(w) = \alpha \restriction A(w)$ for all $w \subseteq l$ with $|w| \leqq j$,

(iv) $\beta_j(u, |u|)\gamma_j\beta_j(|u|, u)$ has an extension in $\mathrm{aut}(M'_2)$ for all $u \subseteq k$ with $|u| \leqq n$.

Induction step. Suppose $j > 0$ and that we have γ_{j-1} satisfying (i)–(iv). By (iv) for any $w \subseteq k$ with $|w| = j$ there exists $\delta_w \in \mathrm{aut}(M'_2)$ such that

$$\beta_{j-1}(w, |w|)\gamma_{j-1}\beta_{j-1}(|w|, w) \subseteq \delta_w.$$

Let

$$\gamma_j \restriction A(w) = (\alpha(|w|, w)\delta_w\alpha(w, |w|)) \restriction A(w).$$

For the rest γ_j is defined by $\gamma_j \supseteq \gamma_{j-1}$. Note that if $w \subseteq l$ and $|w| = j$ we have $\gamma_{j-1} \restriction A(w, j-1) \subseteq \alpha$, whence

$$\beta_{j-1}(w, |w|)\gamma_{j-1}\beta_{j-1}(|w|, w) \subseteq \beta(w, |w|)\alpha\beta(|w|, w).$$

In this case we take $\delta_w \in \mathrm{aut}(M'_2)$ extending $\beta(w, |w|)\alpha\beta(|w|, w)$. This ensures that $\gamma_j \restriction A(w) = \alpha \restriction A(w)$. It is clear that γ_j satisfies (i)–(iii). To verify (iv) we use exactly the same technique as was used in the definition of the maps $\alpha(w, \sigma(w))$ for $w \subseteq l$, $|w| \leqq n$, $\sigma: w \to l$, and $\sigma \uparrow$. We omit the details.

Let $\beta = \gamma_n$. Then $\beta \in \mathrm{perm}(N_k)$ and $\beta(A(w)) = A(w)$. From (iv) $\beta(u, |u|)\beta\beta(|u|, u)$ has an extension $\xi_u \in \mathrm{aut}(M'_2)$ for all $u \subseteq k$ with $|u| \leqq n$. We can choose $\pi \in \mathrm{perm}(l)$ such that $\pi \restriction |u|$ is the identity and $\gamma_\pi\xi_u(I'_{1,2}) = I'_{1,2}$. Then $\gamma_\pi\xi_u$ and ξ_u agree on $A(|u|) \cup A_0(|u|)$, and $\gamma_\pi\xi_u(M'_1) = M'_1$. Hence $\beta(u, |u|)\beta\beta(|u|, u)$ has an extension in $\mathrm{aut}(M'_1)$. Thus $\beta \in G$. Since $\alpha \subseteq \beta$ we are done.

Claim 9. *N_k is a homogeneous extension of N_l.*

Proof. Let $\beta \in \mathrm{aut}(N_k)$ and $\pi \in \mathrm{perm}(k)$ be the one such that $\beta(A(w)) = A(\pi(w))$ for all $w \subseteq k$ with $|w| \leqq n$. Choose $\sigma, \rho \in \mathrm{perm}(k)$ such that

$\sigma\pi(\{0,\dots,l-1\})=\{0,\dots,l-1\}$, $\sigma\restriction(l\cap\pi(l))$ is the identity, and $\rho=(\sigma\pi)^{-1}$. Then $\gamma_\sigma\beta(N_l)=N_l$, $\gamma_\sigma\restriction A(w)$ is the identity for all $w\subseteq l\cap\pi(l)$, $\gamma_\rho\gamma_\sigma\beta(A(w))=A(w)$ for all $w\subseteq k$ with $|w|\leqq n$. Since the γ_π ($\pi\in\operatorname{perm}(k)$) are a system of representatives for the cosets of G we have $\gamma_\rho\gamma_\sigma\beta\in G$. Comparing Claim 5 and the definition of G we have $(\gamma_\rho\gamma_\sigma\beta)\restriction M_2'\in\operatorname{aut}(M_2')$. Also from the definitions $\gamma_\rho\restriction M_2'=\gamma_{\rho|l}$. Hence

$$(\gamma_\sigma\beta)\restriction M_2'=(\gamma_{\rho|l})^{-1}((\gamma_\rho\gamma_\sigma\beta)\restriction M_2')\in\operatorname{aut}(M_2').$$

Let $\alpha=(\gamma_\sigma\beta)\restriction M_2'$. Since $\alpha(m_0')=\beta(M_0')=M_0'$, $\alpha\in\operatorname{aut}(N_l)$. Also, if $w\subseteq l$, $\pi(w)\subseteq l$, and $|w|\leqq n$, then $\gamma_\sigma\restriction A(\pi(w))$ is the identity. Thus $\alpha\restriction A(w)=\beta\restriction A(w)$ if $w\subseteq l$, $\pi(w)\subseteq l$, and $|w|\leqq n$. Hence $\alpha\in\operatorname{aut}(N_l)$ and agrees with β on $N_l\cap\beta^{-1}(N_l)$. Together with Claim 8 this is sufficient.

Since $a(M')\leqq m$ and every m-tuple of N_k is conjugate to one of N_l we have a canonical way of extending the interpretation of L on N_l (which is given since $N_l=(M_2',\{I_{0,2}'\})$) to N_k. Thus the notation $N_k\restriction L$ is unequivocal.

CLAIM 10. *Let $M\in\boldsymbol{P}(L)$ be ω-stable, I be a nice indiscernible set attached to M, $|I|=l+k\geqq 2l$, and M be obtained from M_2' by stretching I_2^* to I. There is an isomorphism ξ from M onto $N_k\restriction L$ which is the identity on M_2'.*

PROOF. Let $I_0,I_2\subseteq I$ correspond to $I_{0,2}'$, I_2^*. Let $b_0,\dots,b_{k-1}$ be an enumeration of $I-I_0$ such that b_i corresponds to $b_{i,2}$ for each $i<l$. For $w\subseteq k$ with $|w|\leqq n$ let $A(w)$ be defined as above for N_k. In M define

$$A'(w)=\{a\in M:\{b_i:i\in w\}\subseteq\operatorname{cl}^I(\{a\})\subseteq I_0\cup\{b_i:i\in w\}\}.$$

Of course $A'(w)=A(w)$ if $w\subseteq l$. Let

$$A_0'(w)=\bigcup\{A'(u):u\subsetneqq w\}\quad(w\subseteq k,|w|\leqq n),$$

$$A_1'(w,v)=\bigcup\{A'(u):u\subseteq v,|u|\leqq|v|,u\neq w\}\quad(w\subseteq v\subseteq k,|v|\leqq l).$$

The following are analogues of Claims 1 and 2.

SUBCLAIM 1. *Every $\alpha\in\operatorname{aut}(M_2')$ has an extension in* $\operatorname{aut}(M)$.

SUBCLAIM 2. *If $w\subseteq v\subseteq k$ and $|v|\leqq l$ then*

$$\operatorname{aut}_M(A'(w),A_0'(w))=\operatorname{aut}_M(A'(w),A_1'(w,v)).$$

Subclaim 1 is clear from 3.4 since M_2' is a full substructure. By symmetry it is enough to establish Subclaim 2 for the case $v\subseteq l$. But this case holds because we can confine attention to automorphisms of M which restrict to automorphisms of

N_l — this comes from the niceness of I — and then the conclusion is immediate from Claim 2.

Returning to the proof of Claim 10 we construct a family

$$\alpha'(w,\sigma(w)) : A'(w) \to A'(\sigma(w)) \qquad (w \subseteq k, |w| \leqq n, \sigma : w \to k, \text{ and } \sigma \uparrow)$$

of bijections such that for all such w and σ

$$\alpha'(w,\sigma(w)) = \alpha'(\sigma(w), |w|)^{-1}\alpha'(w, |w|)$$

and for all $v \subseteq k$ with $|v| \leqq l$, and $\sigma : v \to k$ with $\sigma \uparrow$

$$\beta'(v,\sigma(v)) = \bigcup\{\alpha'(u,\sigma(u)) : u \subseteq v, |u| \leqq n\}$$

can be extended to an automorphism of M. If $w \subseteq l$, set $\alpha'(w,|w|) = \alpha(w,|w|)$. The rest of the construction is like that of $\alpha(w,\sigma(w))$ for M'_2. We omit the details.

The desired isomorphism from M onto $N_k \restriction L$ is

$$\gamma = \bigcup\{\alpha(|w|, w)\alpha'(w,|w|) : w \subseteq k, |w| \leqq n\}.$$

If not, since the relation symbols of L have arity $\leqq m$ there exists $w \subseteq k$ such that $|w| = l$ and the isomorphism fails on $B = \{A(u) : u \subseteq w, |u| \leqq n\}$. Let $\sigma : w \to l$ and $\sigma \uparrow$, and $\pi \in \text{perm}(k)$ extend σ. Then for $u \subseteq w$ with $|u| < n$ we have

$$\begin{aligned}\gamma \restriction A(u) &= \alpha(|u|, u)\alpha'(u, |u|)\\ &= \alpha(\sigma(u), u)\alpha(|u|, \sigma(u))\alpha'(\sigma(u), |u|)\alpha'(u, \sigma(u))\\ &= \alpha(\sigma(u), u)\alpha'(u, \sigma(u))\\ &= (\gamma_\pi)^{-1}\beta'(w, l)) \restriction A(u).\end{aligned}$$

Thus γ and $(\gamma_\pi)^{-1}\beta'(w,l)$ agree on B. But $(\gamma_\pi)^{-1}\beta'(w,l)$ is an L-isomorphism, because $\beta(w,l)$ has an extension in aut(N_k) and $\beta'(w,l)$ has an extension in aut(M). This completes the proof of the claim.

Looking at the construction of N_k we see that $N_l, N_{l+1}, \ldots$ can be constructed so as to form a chain, i.e. so that if $l \leqq j < k$ and $u, v \subseteq j$ with $|u| = |v| \leqq n$, then $A(u)$, $\alpha(u,v)$, and $\beta(u,v)$ will have the same meaning in N_k as in N_j. Also, if $u \subseteq j$, $\pi \in \text{perm}(j)$, $\sigma \in \text{perm}(k)$, and $\pi \restriction u = \sigma \restriction u$, then $\gamma_\pi \restriction A(u) = \gamma_\sigma \restriction A(u)$. Henceforth we assume that $N_l, N_{l+1}, \ldots$ have been so constructed.

Let M_l denote $N_l \restriction L$. Then $M_l, M_{l+1}, \ldots$ is an ascending chain of L-structures.

We recall some notation from our proof plan. From its behaviour in M'_2 we know that $\psi(x_0, x_1)$ defines $E_k \in \mathscr{E}(M_k \times M_k)$. Let I_k denote $(M_k \times M_k)/E_k$. For $l \leqq j < k$ let $I_{j,k}$ denote the range of the canonical embedding of I_j into I_k. Let J_k be the image of $I'_{0,2}$ under the canonical embedding of $I^*_2 = I_l$ into I_k. From Claim 8, J_k is indiscernible in N^*_k. Also J_k is defined by χ.

CLAIM 11. *Let* $l \leqq j \leqq k$, $\alpha \in \operatorname{aut}(M_j)$, *and* $\alpha(J_j) = J_j$. *There exists* $\beta \in \operatorname{aut}(N_k)$ *such that* $\beta \supseteq \alpha$.

PROOF. Let α satisfy the hypothesis and $\pi \in \operatorname{perm}(j)$ be the unique one such that $\alpha(A(w)) = A(\pi(w))$ for all $w \subseteq j$ with $|w| \leqq n$. Let $\sigma \in \operatorname{perm}(k)$ extend π. Then $\gamma_\pi \subseteq \gamma_\sigma$, $(\gamma_\pi)^{-1}(\alpha(A(w)) = A(w)$ for all $w \subseteq j$ with $|w| \leqq n$, $(\gamma_\pi)^{-1}\alpha \in \operatorname{aut}(M_j)$ and $(\gamma_\pi)^{-1}\alpha(J_j) = J_j$. Thus it is sufficient to prove $(\gamma_\pi)^{-1}\alpha$ has an extension in $\operatorname{aut}(N_k)$, i.e. we can assume $\alpha(A(w)) = A(w)$ for all $w \subseteq j$ with $|w| \leqq n$.

Since $N_l = (M'_2, \{I'_{0,2}\})$ and $L = L(M'_2)$, $\alpha \restriction M'_2 \in \operatorname{aut}(N_l)$. Hence $\alpha \restriction M'_2$ has an extension in $\operatorname{aut}(N_k)$ by Claim 8. Therefore if $w \subseteq l$ and $|w| = n$, $\alpha \restriction (A(w) \cup A_0(w))$ has an extension in $\operatorname{aut}(N_k)$. Since the indiscernibility of $I_j - J_j$ in N^*_j is witnessed by the γ_π ($\pi \in \operatorname{perm}(j)$) which extend to automorphisms of N_k, $\alpha \restriction (A(w) \cup A_0(w))$ has an extension in $\operatorname{aut}(N_k)$ for all $w \subseteq j$ with $|w| = n$. For the rest we follow the same line as in the proof of Claim 8.

Claim 11 will be used later when we show that any $\alpha \in \operatorname{aut}(M_j)$ has an extension $\beta \in \operatorname{aut}(M_k)$ whenever $l \leqq j \leqq k$. For the next claim we require only the weak version of Claim 11 represented by Claim 8.

CLAIM 12. *If* $l \leqq k \leqq \omega$ *then* $N_k, M_k \in \boldsymbol{C}$ *and both are* ω*-stable.*

PROOF. Since M_k is a reduct of N_k it is sufficient to prove that $N_k \in \boldsymbol{C}$ and that N_k is ω-stable. Since $M' \in \boldsymbol{H}$ so is $N_l = (M'_2, \{I'_{0,2}\})$. From Claim 8 every $\alpha \in \operatorname{aut}(N_l)$ extends to $\beta \in \operatorname{aut}(N_k)$ and from Claim 9, N_k is a homogeneous extension of N_l. In the course of the proof we shall often use these facts without explicit mention. In particular, note that 0-definable equivalence relations on N_l extend to N_k, that indiscernibility and strong indiscernibility lift from N_l to N_k, and that $\operatorname{cl}^I(A)$ and $\operatorname{scl}^I(A)$ for A, $I \subseteq N^*_l$ are the same with respect to N_k. By scl^I we mean the same as cl^I but with strong indiscernibility instead of indiscernibility.

First we show that N_k is $\aleph_0$-categorical. For proof by contradiction consider the least $i < \omega$ such that infinitely many $(i+1)$-types are realized in N_k. There exist $a_0, \dots, a_{i-1}$ in N_k such that infinitely many 1-types over $A = \{a_0, \dots, a_{i-1}\}$ are realized in N_k. Let $a_j \in A(w_j)$, $j < i$, and $w \subseteq k$ satisfy $|w| = n$ and either

$w \cap \bigcup\{w_j : j < i\} = \emptyset$ or $k = w \cup \bigcup\{w_j : j < i\}$. Using the γ_π ($\pi \in \mathrm{perm}(k)$) we see that a 1-type over A realized in N_k is realized in $A(u)$ for some $u \subseteq w \cup \bigcup\{w_j : j < i\}$. We can fix $u \subseteq k$ with $|u| \leqq n$ such that infinitely many 1-types over $\{a_0, \ldots, a_{i-1}\}$ are realized in $A(u)$. Further we can assume $u \subseteq l$ so that $A(u) \subseteq N_l$. Since $N_l \in \boldsymbol{H}$ only a finite number of 1-types over $\emptyset$ are realized in $A(u)$.

Let $E^\# \in \mathscr{E}(M')$ be defined by

$$E^\# = \{\langle c_0, c_1\rangle \in (M')^2 : \mathrm{cl}^I(\{c_0\}) = \mathrm{cl}^I(\{c_1\})\}.$$

The L-formula $\nu(x_0, x_1)$ defining $E^\#$ defines $E_k^\# \in \mathscr{E}(N_k)$ and $A(u)$ is the union of a finite number of $E^\#$-classes which are also $E_k^\#$-classes. Let $C \subseteq A(u)$ be minimal with respect to the properties: C is an equivalence class of a 0-definable equivalence relation, and infinitely many 1-types over A are realized in C. Note that C is transitive by its minimality.

By 9.2, attached to N_l and thus also to N_k is a $\{C\}$-definable set $B = I_0 \cup \cdots \cup I_{h-1}$ such that $I_0, \ldots, I_{h-1}$ are infinite and strongly mutually indiscernible over $\{C\}$ in N_l, and $\mathrm{acl}(\{a\}) \cap B \neq \emptyset$ for all $a \in C$.

Case 1. $\mathrm{cl}^B(A)$ exists in N_k, i.e. there exists a least $D \subseteq B$ such that $|D| < \aleph_0$ and $I_0 - D, \ldots, I_{h-1} - D$ are mutually indiscernible over $D \cup A$ in N_k. In N_l, and hence also in N_k, C is partitioned into a finite number of 1-types over D. Let P be one of these 1-types such that over A infinitely many 1-types are realized in P. Let

$$E^+ = \{\langle c_0, c_1\rangle \in C^2 : \mathrm{cl}^B(\{c_0\}) = \mathrm{cl}^B(\{c_1\})\}$$

and $E_P^+ = E^+ \cap P^2$. Since $D = \mathrm{cl}^B(A)$, every 1-type over A realized in one E_P^+-class is approximated as closely as we like in any other E_P^+-class. Hence in every E_P^+-class infinitely many 1-types over A are realized. Thus in some E^+-class the same is true. However, since B is $\{C\}$-definable each E^+-class is an equivalence class of a member of $\mathscr{E}(N_l)$. This contradicts the choice of C and so finishes this case.

Case 2. Otherwise. This breaks up into a number of subcases which we shall not treat in detail. Let

$$H = \{\pi \in \mathrm{perm}(I_0 \cup \cdots \cup I_{h-1}) : \pi(I_j) = I_j \text{ for all } j < h\}.$$

Each $\pi \in H$ can be extended to $\alpha_\pi \in \mathrm{aut}(N_l)$ and hence to $\beta_\pi \in \mathrm{aut}(N_k)$. From the nonexistence of $\mathrm{cl}^B(A)$ in N_k it follows that $|\{\beta_\pi(A) : \pi \in H\}| = 2^{\aleph_0}$ which contradicts $|N_k| = \aleph_0$.

To give the flavour of the rest of the argument we treat the subcase in which $\mathrm{cl}^{I_0}(A)$ does not exist. If $H_0 = \mathrm{aut}_{N_k}(I_0, A)$ has infinitely many orbits we see at once that $|\{\beta_\pi(A) : \pi \in H\}| = 2^{\aleph_0}$. By naming a finite number of elements of I_0 if necessary we reduce to the case in which H_0 is transitive. Now H_0 is not dense in perm(I_0), i.e. some finite one-to-one map $\sigma \subseteq I_0^2$ has no extension in H_0, since $\mathrm{cl}^{I_0}(A)$ does not exist. The contradiction now comes from the fact that any nondense transitive subgroup of perm(ω) has $2^{\aleph_0}$ conjugates.

We now turn to the proof of ω-stability. From [12, III, 5.17] it is sufficient to prove that N_k is superstable since we already have $\aleph_0$-categoricity. Call definable $A \subseteq N_k$ *unsuperstable in* N_k if there are $A_\eta \subseteq N_k$ $(\eta \in {}^{<\omega}\omega)$ such that

(i) for $i < \omega$, the sets A_η with $l(\eta) = i$ are uniformly definable in N_k,

(ii) if ξ, $\eta \in {}^{<\omega}\omega$ and $\xi \subseteq \eta$, then $A_\xi \supseteq A_\eta$,

(iii) if $\eta \in {}^{<\omega}\omega$ and $\Xi \subseteq \{\xi \in {}^{<\omega}\omega : \xi \not\subseteq \eta$ and $\eta \not\subseteq \xi\}$, then

$$A \cap A_\eta \not\subseteq \bigcup\{A_\xi : \xi \in \Xi\}.$$

Since N_k is $\aleph_0$-categorical, taking $A = N_k$ we have a condition equivalent to the usual definition of the unsuperstability of N_k. If $A = \bigcup\{A_j : j < i\}$ is a finite union of definable sets of N_k and A is unsuperstable in N_k, so is A_j for some $j < i$.

Towards a contradiction let N_k be unsuperstable. We assume $k < \omega$ because the same kind of argument can also be used to eliminate the case $k = \omega$. Then $A(u)$ is unsuperstable for some $u \subseteq k$ with $|u| \leqq n$, and as before we can suppose $u \subseteq l$ so that $A(u) \subseteq N_l$. Let $C \subseteq A(u)$ be minimal with respect to the properties: C is an equivalence class of a 0-definable equivalence relation, and C is unsuperstable in N_k. Note that C is transitive.

Let $B = I_0 \cup \cdots \cup I_{h-1}$ be a $\{C\}$-definable set attached to N_l having the same relation to C as before. By Case 2 of the argument for $\aleph_0$-categoricity $\mathrm{cl}^B(A)$ exists in N_k for every finite $A \subseteq N_k$. Let $D \subseteq B$ be a maximal set such that $C_D = \{a \in C : D \subseteq \mathrm{cl}^B(\{a\})\}$ is unsuperstable in N_k. Let

$$E^+ = \{\langle c_0, c_1\rangle \in C^2 : \mathrm{cl}^B(\{c_0\}) = \mathrm{cl}^B(\{c_1\})\}$$

and $E_D^+ = E^+ \cap C_D^2$. Let $\{A_\eta : \eta \in {}^{<\omega}\omega\}$ be a family witnessing the unsuperstability of C_D.

Case 1. There exist $i < \omega$, $\eta \in {}^i i$, and $A^+ \subseteq N_k$ such that

(i) $|A^+| < \aleph_0$ and A_ξ is A^+-definable in N_k for all $\xi \in {}^{\leq i}i$, and

(ii) for every E_D^+-class C^+ such that for all $a \in C^+$,

$$(\mathrm{cl}^B(\{a\}) - D) \cap \mathrm{cl}^B(A^+ \cup D) = \varnothing,$$

we have

$$A_\eta \cap C^+ \subseteq \bigcup\{A_\xi : \xi \in {}^{\leq i}i,\ \xi \not\subseteq \eta\}.$$

Then since the E_D^+-classes cover C_D we have

$$(A_\eta \cap C_D) - \bigcup\{A_\xi : \xi \in {}^{\leq i}i,\ \xi \not\subseteq \eta\} \subseteq \bigcup\{C_{D\cup\{b\}} : b \in \mathrm{cl}^B(A^+ \cup D) - D\}.$$

But the set on the left is unsuperstable in N_k, whence $C_{D\cup\{b\}}$ is unsuperstable for some $b \in \mathrm{cl}^B(A^+ \cup D) - D$. This contradicts the choice of D finishing Case 1.

Case 2. Otherwise. With every E_D^+-class C^+ is associated a finite set $B(C^+) \subseteq B - D$ such that $\mathrm{cl}^B(\{a\}) = D \cup B(C^+)$ for all $a \in C^+$. Since C is transitive, $|B(C^+)|$ is the same for all $C^+ \in C_D/E_D^+$. Two E_D^+-classes are conjugate over $D \cup \{I_0, \dots, I_{h-1}\}$ if $|B(C_0^+) \cap I_i)| = |B(C_1^+) \cap I_i|$ for all $i < h$.

Call $\mathscr{C} = \{C_0^+, \dots, C_{e-1}^+\} \subseteq C_D/E_D^+$ an *independent set of representatives* if

(i) every conjugacy class of E_D^+-classes over $D \cup \{I_0, \dots, I_{h-1}\}$ is represented exactly once in $\mathscr{C}$, and

(ii) $B(C_i^+) \cap B(C_j^+) = \varnothing$ for all $i < j < e$.

Note that any two independent sets of representatives are conjugate over $\{C\} \cup D$. The failure of Case 1 says that for every $i < \omega$, if $A^+ \subseteq N_k$ is a finite set containing all the parameters needed to define the sets A_ξ ($\xi \in {}^{\leq i}i$) and $\mathscr{C}$ is an independent set of representatives with

$$\bigcup\{B(C^+) : C^+ \in \mathscr{C}\} \cap \mathrm{cl}^B(A^+ \cup D) = \varnothing,$$

then for all $\eta \in {}^i i$

$$A_\eta \cap \bigcup \mathscr{C} \not\subseteq \{A_\xi : \xi \in {}^{\leq i}i,\ \xi \not\subseteq \eta\}.$$

By the compactness theorem, if $\mathscr{C}$ is an independent set of representatives then $\bigcup \mathscr{C}$ is unsuperstable in N_k. Hence some E_D^+-class C^+ is unsuperstable. Therefore some E^+-class is unsuperstable. This contradicts the choice of C and completes the proof of the claim.

Recall the definition of I_k from just before Claim 11.

CLAIM 13. *I_k is a nice indiscernible set attached to M_k and $i(I_k) = n$.*

PROOF. From Claim 8 every $\alpha \in \mathrm{aut}(N_l)$ has an extension $\beta \in \mathrm{aut}(N_k)$. If $a \in M_2'$, $\mathrm{cl}^{J_l}(\{a\})$ exists in N_l and so $\mathrm{cl}^{J_k}(\{a\})$ exists in N_k. Since every $a \in N_k$ is conjugate in N_k to some $a' \in M_2'$, $\mathrm{cl}^{J_k}(\{a\})$ exists for every $a \in N_k$. Let M_k^+ denote the L-substructure of N_{k+l} whose universe is

$$\{\alpha \in N_{k+l} : \mathrm{cl}^{J_{k+l}}(\{\alpha\}) = \varnothing\}.$$

Note that M_k^+ is 0-definable in N_{k+l}. Let $I_k^+ \subseteq (M_k^+)^*$ be the set defined by ψ. The canonical embedding $\tau : I_k^+ \to I_{k+l} - J_{k+l}$ is a bijection. Since $\alpha \upharpoonright M_k^+ \in \text{aut}(M_k^+)$ for all $\alpha \in \text{aut}(N_{k+l})$, $\tau(\text{cl}^{I_k^+}(\{a\}))$ exists for all $a \in M_k^+$ and

$$\tau(\text{cl}_{M_k^+}^{I_k^+}(\{a\})) \subseteq \text{cl}_{N_{k+l}}^{I_{k+l}-J_{k+l}}(\{a\}).$$

In fact we have equality because if not the inclusion would be proper when $k = 0$, contradiction. By construction of N_{k+l}, $I_{k+l} - J_{k+l}$ is a nice indiscernible set attached to N_{k+l}. Since $\text{cl}_{M_k^+}^{I_k^+}$ corresponds to $\text{cl}_{N_{k+l}}^{I_{k+l}-J_{k+l}}$, I_k^+ is a nice indiscernible set attached to M_k^+.

Let $I_3' \subseteq I'$ be chosen such that $I_2' \subseteq I_3'$ and $|I_3'| = 3l$. Let M_3' be the homogeneous substructure of M' with universe $\{a \in M' : \text{cl}^{I'}(\{a\}) \subseteq I_3'\}$. Let I_3^* and $I_{i,3}'$ $(i < 3)$ be defined by analogy with I_2^* and $I_{i,2}'$ $(i < 2)$. By Claim 10 we can suppose that $N_{2l} \upharpoonright L = M_3'$. By Claim 11 with $j = k = 2l$ it then follows that $N_{2l} = (M_3', I_{0,3}')$. Also M_l^+ is the homogeneous substructure of M_3' with universe $\{a \in M_3' : \text{cl}^{I'}(\{a\}) \subseteq I_3' - I_0'\}$. If $\alpha \in \text{aut}(M_l^+)$ then there exists $\beta \supseteq \alpha$ such that $\beta \in \text{aut}(M_3')$, and clearly we must have $\beta(I_{0,3}') = I_{0,3}'$. Thus each $\alpha \in \text{aut}(M_l^+)$ extends to N_{2l} and hence to N_{k+l} by Claim 11. Hence each $\alpha \in \text{aut}(M_l^+)$ extends to $\beta \in \text{aut}(M_k^+)$ for all $k \geqq l$.

Let $I_{l,k}^+$ denote the range of the canonical embedding of I_l^+ into I_k^+. Then

$$M_l^+ = \{a \in M_k^+ : \text{cl}^{I_k^+}(\{a\}) \subseteq I_{l,k}^+\}.$$

Hence M_l^+ is the universe of a homogeneous substructure of M_k^+ and since automorphisms of M_l^+ extend to M_k^+, M_l^+ is a homogeneous substructure. Therefore M_l^+ is obtained from M_k^+ by shrinking I_k^+ to I_l^+.

Since $|I_{2,3}'| = |I_3^* - I_{0,3}'| = 2l$ there exists $\alpha \in \text{aut}(M_3')$ such that $\alpha(M_2') = M_l^+$. Thus M_2' and M_l^+ are isomorphic by an automorphism which matches I_2^* and I_l^+. It now follows by Claim 10 that there is an isomorphism of M_k^+ onto $N_k \upharpoonright L = M_k$. Hence I_k is a nice indiscernible set attached to M_k because I_k^+ is a nice indiscernible set attached to M_k^+. We have $i(I_k) = n$, because $i(I_k^+) = n$ by inspection of N_{k+l}.

Claim 14. *If $l \leqq j < k$, then*

$$M_j = \{a \in M_k : \text{cl}_{M_k}^{I_k}(\{a\}) \subseteq I_{j,k}\}.$$

Proof. Left to the reader.

Claim 15. *If $l \leqq j < k$, then every $\alpha \in \text{aut}(M_j)$ extends to $\beta \in \text{aut}(M_k)$.*

Proof. From Claim 14 every permutation of I_j is induced by some $\gamma \in$

$\mathrm{aut}(M_k)$ such that $\gamma(M_j) = M_j$. From Claim 11 if $\alpha \in \mathrm{aut}(M_j)$ and $\alpha(J_j) = J_j$ then α extends to $\beta \in \mathrm{aut}(M_k)$. Therefore if $\alpha \in \mathrm{aut}(M_j)$ and α fixes any subset of I_j of power l then α extends to $\beta \in \mathrm{aut}(M_k)$. But every $\alpha \in \mathrm{aut}(M_j)$ can be expressed as the product of automorphisms each fixing at least l elements of I_j, so the desired conclusion follows.

At last we are ready to finish. From Claims 14 and 15 for $l \leqq j < k$, M_j is a homogeneous substructure of M_k and is obtained from M_k by shrinking I_k to I_j. From Claim 10 there is an isomorphism between M' and $M_{|I'|-l}$ which is the identity on M'_2. So we can identify M' with $M_{|I'|-l}$. Let $N \in \boldsymbol{P}(L)$ be an ω-stable structure with a nice indiscernible set J attached to it such that $|J| = k \geqq |I'|$ and N is obtained from M' by stretching I' to J. Then N is obtained from M'_2 by stretching I^*_2 to J. From Claim 10 there is an isomorphism ξ from N onto M_{k-l}. For some $\pi \in \mathrm{perm}(k-l)$ we have $\gamma_\pi\xi(M') = M'$. By Claim 15, $(\gamma_\pi\xi)\restriction M'$ extends to $\beta \in \mathrm{aut}(M_{k-l})$. Thus $\beta^{-1}\gamma_\pi\xi$ is an isomorphism of N onto M_{k-l} which is the identity on M'. This completes the proof of the lemma

Question 13.4. Is $M_k \in \boldsymbol{H}(L)$ for all $k \geqq l$?

Question 13.5. Is $M_k \in \boldsymbol{H}$ for all $k \geqq l$?

Question 13.6. Can there exist a non-ω-stable structure $M \in \boldsymbol{P}(L)$ and a nice indiscernible set I attached to M such that M is obtained from M' by stretching I' to I?

14. Stretching a structure

In §12 we described a method of shrinking structures. Here we shall show that under certain conditions the shrinking process is uniquely reversible. Our best result in this direction is as follows. Let $M, N \in \boldsymbol{H}(L)$ and N be obtained by shrinking M. Let

$$m = \inf\{i : (\exists \phi \in \Phi(M))[d_N(\phi) = i < d_M(\phi)]\}.$$

If m is sufficiently large compared with $l(L)$ and $r(N)$, then the isomorphism type of M over N is uniquely determined by $\langle N, d_M \rangle$. We have to work up to this in easy stages.

In the sequel from §13 we require only the uniqueness part of 13.3 which is essentially Claim 10 of the proof of 13.3. (The proof of 13.3 could have been written so that M' appeared instead of M_2 in Claim 10 if we had been satisfied to leave the matter there.) The uniqueness is conveniently stated as follows.

LEMMA 14.1. *Let L be a finite relational language, $M', M_j \in \boldsymbol{H}(L)$, $\psi \in \Psi(M') \cap \Psi(M_j)$ $(j<2)$, $d_{M'}(\psi) < d_{M_0}(\psi) = d_{M_1}(\psi)$, $i_{M_0}(\psi) = i_{M_1}(\psi) = i_{M'}(\psi)$, and $M' = M_j(\psi, d_{M'}(\psi))$ $(j<2)$. If $i_{M'}(\psi) \cdot \max\{a(M'), 8\} \leqq d_{M'}(\psi)/3 < \omega$, then there is an isomorphism $\xi : M_0 \rightarrow M_1$ with $M' \subseteq \mathrm{fix}(\xi)$.*

PROOF. In 13.3 let $k = d_{M_0}(\psi) = d_{M_1}(\psi)$ and M be as in the conclusion. Then M_j has the same isomorphism type over M' as M $(j<2)$. Therefore M_0 and M_1 have the same isomorphism type over M'.

The next step is to show that if $M' \in \boldsymbol{H}(L)$, $\mathcal{F} \in \boldsymbol{F}(M')$, $|\mathcal{F}| < \omega$, and $d(\mathcal{F})$ is large enough compared with $l(L)$, $r(M')$, and $|\mathcal{F}|$, then there is at most one way of "stretching" $\mathcal{F}$ while keeping the structure homogeneous for the same finite relational language.

LEMMA 14.2. *Let L be a finite relational language, $M', M_j \in \boldsymbol{H}(L)$, $\boldsymbol{\phi} \in \Phi(M') \cap \Phi(M_j)$ $(j<2)$, $m < d_{M_0}(\boldsymbol{\phi}) = d_{M_1}(\boldsymbol{\phi})$, and $M' = M_j(\boldsymbol{\phi}, m)$ $(j<2)$. Let $\mathcal{F} \in \boldsymbol{F}(M')$ be associated with $\boldsymbol{\phi}$. If m is sufficiently large compared with $l(L)$, $r(M')$, and $|\mathcal{F}|$, then there is an isomorphism $\xi : M_0 \rightarrow M_1$ with $M' \subseteq \mathrm{fix}(\xi)$.*

PROOF. From 12.2, $l(M_j) = l(M')$ and $r(M_j) = r(M')$ $(j<2)$. Let $\mathcal{F}_j \in \boldsymbol{F}(M_j)$ be associated with $\boldsymbol{\phi}$ $(j<2)$. Let $N' = (M', \mathcal{F})$ and $N_j = (M_j, \mathcal{F}_j)$ $(j<2)$. From 5.2, $N, N_j \in \boldsymbol{H}$ $(j<2)$ and $l(N')$, $r(N')$, $l(N_j)$, $r(N_j)$ are bounded in terms of $l(L)$, $r(M')$ and $|\mathcal{F}|$. From 12.3 (ii) the canonical embedding of $\mathcal{F}$ into $\mathcal{F}_j$ is a bijection. Looking at automorphisms it is easy to see that N' is a homogeneous substructure of N_j $(j<2)$. Also for all $a \in N'$

$$\mathrm{cl}^{\mathcal{F}}_{M'}(\{a\}) = \bigcup\{\mathrm{cl}^{I}_{N'}(\{a\}); I \in \mathcal{F}\}$$

and similarly for $\mathcal{F}_j$, M_j, and N_j instead of $\mathcal{F}$, M', and N' respectively.

If m is sufficiently large compared with $l(N_j)$ then $l(N') = l(N_j)$. Thus we may suppose that N', N_0, N_1 are all homogeneous for the same finite relational language L^+ extending L.

Let $I_0, \ldots, I_{e-1}$ be an enumeration of $\mathcal{F}$. For $i < e$, I_i is a nice indiscernible set attached to N'. Let ψ_i be the nice formula for N' associated with I_i. For $j<2$ define sequences $\langle N_{j,i} : i < e\rangle$ of structures in $\boldsymbol{H}(L^+)$ as follows. Let $N_{j,0} = N_j$ and, for $i < e$, let $N_{j,i+1} = N_{j,i}(\psi_i, m)$. More precisely let $N_{j,i+1}$ be the homogeneous substructure of $N_{j,i}$ whose universe is

$$\{a \in N_{j,i} : \mathrm{cl}^{I_{j,i}}_{N_{j,i}}(\{a\}) \subseteq I'_{j,i}\},$$

where $I_{j,i} \subseteq N_{j,i}$ is the indiscernible set defined by ψ_i and $I'_{j,i}$ is the range of the canonical embedding of I_i onto $I_{j,i}$.

From the general properties of shrinking developed in §12, since m is large compared with $l(N_j)$ the inductive definition just given is sound. Moreover $N_{j,e} = N'$ ($j < 2$). We have bounded the language L^+, whence $i(I)$ is bounded for any 0-definable indiscernible set I attached to a structure in $\boldsymbol{H}(L^+)$. From 12.3 (vi) for all i, $k < e$ and $j < 2$

$$i_{N_{j,i+1}}(\psi_i) = i_{N_{j,i}}(\psi_i) = i_{N'}(\psi_i).$$

By 12.3 (iii) and (v) for all $i < e$ and $j < 2$

$$d_{N_{j,i+1}}(\psi_i) = m < d_{N_{j,i}}(\psi_i) = d_{M_j}(\boldsymbol{\phi}).$$

By hypothesis $d_{M_0}(\boldsymbol{\phi}) = d_{M_1}(\boldsymbol{\phi})$.

By backwards induction on i using 14.1 for the induction step we see that for all $i \leqq e$ there is an isomorphism $\xi_i : N_{0,i} \to N_{1,i}$ with $N' \subseteq \operatorname{fix}(\xi_i)$. Letting $i = 0$ we have an isomorphism $\xi : N_0 \to N_1$ with $N' \subseteq \operatorname{fix}(\xi)$.

We now prove the main result of the section.

THEOREM 14.3. *Let L be a finite relation language, $M, M' \in \boldsymbol{H}(L)$, $\boldsymbol{\phi} \in \Phi(M) \cap \Phi(M')$, $m < d_M(\boldsymbol{\phi})$, and $M' = M(\boldsymbol{\phi}, m)$. If m is sufficiently large compared with $l(L)$ and $r(M')$, then the isomorphism type of M over M' is uniquely determined by $\langle M', \boldsymbol{\phi}, d_M(\boldsymbol{\phi})\rangle$.*

PROOF. For $j < 2$ let M_j satisfy the hypothesis about M for the same $\boldsymbol{\phi}$ and m, where m is large compared with $l(L)$ and $r(M')$. It is sufficient to prove that there is an isomorphism ξ of M_0 onto M_1 with $M' \subseteq \operatorname{fix}(\xi)$.

For $k < \omega$ and $N \in \boldsymbol{H}(L)$ let $\Phi^k(N)$ denote $\{\boldsymbol{\phi} \in \Phi(N) : d_N(\boldsymbol{\phi}) \geqq k\}$. From 11.3 given L we can compute k such that for any $N \in \boldsymbol{H}(L)$ the restriction $\leqq_N \mid \Phi^k(N)$ is transitive and

$$(\leqq_N \cap \geqq_N) \restriction \Phi^k(N) = \approx_N \restriction \Phi^k(N).$$

Fix such k and let $\langle \boldsymbol{\phi}_i : i < n\rangle$ be a sequence of representatives of all the $\approx_{M'}$-equivalence classes in $\Phi^k(M')$ ordered so that $\boldsymbol{\phi}_i <_{M'} \boldsymbol{\phi}_j$ implies $j < i$. Since m is large and $\boldsymbol{\phi} \in \Phi^m(M')$ we can suppose that $\boldsymbol{\phi}_e = \boldsymbol{\phi}$ for some $e < n$. From 12.2 and 12.3, $\langle \boldsymbol{\phi}_i : i < n\rangle$ bears the same relation to M_0, M_1 as it does to M': for $j < 2$, $\langle \boldsymbol{\phi}_i : i < n\rangle$ is a sequence of representatives of all the $\approx_{M_j}$-equivalence classes in $\Phi^k(M_j)$ ordered so that $\boldsymbol{\phi}_i <_{M'} \boldsymbol{\phi}_j$ implies $j < i$. Also, for $i < n$ and $i \neq e$, $d_{M_j}(\boldsymbol{\phi}_i) = d_{M'}(\boldsymbol{\phi}_i)$ ($j < 2$). From 12.2 (iii), $r(M_j) = r(M')$ since m is large compared with $l(L)$ and $r(M')$.

The idea of the proof is as follows. We define sequences $\langle M_{j,i} : 0 \leqq i \leqq e + 1\rangle$

$(j<2)$ in $\boldsymbol{H}(L)$ by induction on i. We let $M_{j,0}=M_j$ and $M_{j,i+1}$ be either $M_{j,i}$ or $M_{j,i}(\phi_i,m_i)$ for certain $m_i<\omega$. By choosing $m_e=m$ we ensure that $M_{0,e+1}\simeq M_{1,e+1}$. By choosing $m_0,\dots,m_{e-1}$ suitably we can apply 14.2 to extend an isomorphism between $M_{0,i+1}$ and $M_{1,i+1}$ to one between $M_{0,i}$ and $M_{1,i}$. Having once got an isomorphism between $M_0=M_{0,0}$ and $M_1=M_{1,0}$ it is easy to find one which is the identity on M'. To ease notation, if $s=d_N(\psi)<\omega$, then we allow $N(\psi,s)$ as another notation for N. With this convention $M_{j,i+1}$ has the form $M_{j,i}(\phi_i,m_i)$, with $m_i\leqq d_{M_{j,i}}(\phi_i)$ and $m_i<\omega$, for all $i\leqq e$ and $j<2$.

To see that $M_{0,e+1}\simeq M_{1,e+1}$ recall from 12.5 that if the order in which one-step shrinkings are performed is changed then the resulting model is unchanged provided the target dimensions are large enough compared with $l(L)$. In the present case if M_j is first shrunk with respect to $\phi_e=\phi$ then $M_{0,1}=M_{1,1}=M'$ after the first step and of course the equality persists.

Now we consider how to choose $m_0,\dots,m_{e-1}$. Let $G:\omega^3\to\omega$ be an increasing function such that m is large enough in 14.2 if $m\geqq G(l(L),r(M'),|\mathcal{F}|)$. From 11.6, if $\mathcal{F}_{j,i}\in\boldsymbol{F}(M_{j,i})$ is associated with ϕ_i then $|\mathcal{F}_{j,i}|$ is bounded in terms of $l(L)$, $r(M')=r(M_{j,i})$, and $\max\{m_h:h<i\}$. Here we are assuming that, if $m_i\neq d_{M_{j,i}}(\phi_i)$, then m_i is chosen large enough compared with $l(L)$ and $r(M')=r(M_j)$ so that by 12.2

$$\Phi(M_{j,i})=\Phi(M'),\quad \approx_{M_{j,i}}=\approx_{M'},\quad \leqq_{M_{j,i}}=\leqq_{M'},\quad r(M_{j,i})=r(M'),$$

and by 12.3

$$d_{M_{j,i+1}}(\psi)=\begin{cases} m_i & \text{if } \psi\approx_{M'}\phi_i,\\ d_{M_{j,i}}(\psi) & \text{otherwise.}\end{cases}$$

We should note by induction on h that

$$d_{M_{j,h}}(\phi_i)=d_{M'}(\phi_i)\qquad (h\leqq i),$$

whence there is no conflict between $M_{0,i}$ and $M_{1,i}$ when we come to choose m_i.

Suppose $m_0,\dots,m_{i-1}$ have been chosen where $i<e$. If there is a possible value for $m_i<d_{M'}(\phi_i)$, "possible" with regard to the need for the consequences of 12.2 and 12.3 just mentioned to hold, such that

$$m_i\geqq G(l(L),r(M'),\max(|F_{0,i}|,|F_{1,i}|)),$$

then let m_i have the least such value. Otherwise let $m_i=d_{M'}(\phi)$ so that $M_{j,i+1}=M_{j,i}$. Notice that $e<n$ is bounded in terms of $l(L)$. Thus the m_i for $i<e$ are bounded uniformly in terms of $l(L)$ and $r(M')$. Thus if m is large enough compared with $l(L)$ and $r(M')$, we have

$$m_e = m \geqq G(l(L), r(M'), \max(|\mathscr{F}_{0,e}|, |\mathscr{F}_{1,e}|))$$

and also the desired consequences of 12.2 and 12.3 mentioned above for the case $i = e$.

By backwards induction on i using 14.2 in the induction step we have $M_{0,i} \simeq M_{1,i}$ for all $i \leqq e$. Hence $M_0 \simeq M_1$. By 12.1 (ii) M_0 and M_1 are prime over M' and so there is an isomorphism ξ of M_0 onto M_1 with $M' \subseteq \mathrm{fix}(\xi)$. This completes the proof of the theorem.

COROLLARY 14.4. *Let L be a finite relational language, M, $N \in \boldsymbol{H}(L)$, and N be obtained by shrinking M. Let*

$$m = \inf\{i : (\exists \boldsymbol{\phi} \in \Phi(M))[d_N(\boldsymbol{\phi}) = i < d_M(\boldsymbol{\phi})]\}.$$

If m is sufficiently large compared with $l(L)$ and $r(N)$, then the isomorphism type of M over N is uniquely determined by $\langle N, d_M \rangle$.

PROOF. By iteration of 14.3.

15. Conclusion

In this section we shall prove the weak versions of Conjectures 0.2, 0.3 mentioned in the Introduction. Recall that $\boldsymbol{H}(L, r)$ denotes $\{M \in \boldsymbol{H}(L) : r(M) \leqq r\}$. The weak form of 0.2 is

THEOREM 15.1. *Let L be a finite relational language and $r < \omega$. There exists $s < \omega$ such that for every $M \in \boldsymbol{H}(L, r)$ there exists $M' \in \boldsymbol{H}(L, r)$ such that $|M'| \leqq s$ and either $M' = M$, or*

(i) *M' is obtained from M by a smooth shrinking, and*

(ii) *if $N \in \boldsymbol{H}(L)$, $d_N = d_M$, and M' is obtained by shrinking N, then there is an isomorphism $\alpha : M \to N$ such that $M' \subseteq \mathrm{fix}(\alpha)$.*

PROOF. Choose the least $m < \omega$ such that

(a) m is large enough compared with $l(L)$ and r for the truth of 14.4 whenever M, $N \in \boldsymbol{H}(L)$ and $r(N) \leqq r$,

(b) any shrinking of $M \in \boldsymbol{H}(L)$ for which all the target dimensions are $\geqq m$ is smooth (see 12.6 (ii)),

(c) $\leqq_M \restriction \Phi^m(M)$ is transitive and

$$(\leqq_M \cap \geqq_M) \restriction \Phi^m(M) = \approx_M \restriction \Phi^m(M)$$

(see 11.3 (i) and (ii)).

Let $M \in \boldsymbol{H}(L, r)$ be given. Obtain $M' \in \boldsymbol{H}(L, r)$ by shrinking M, with one step for each equivalence class in $\Phi^{m+1}(M)$, with target dimension m so that $d_{M'}(\boldsymbol{\phi}) = \min(m, d_M(\boldsymbol{\phi}))$ for all $\boldsymbol{\phi} \in \Phi(M') = \Phi(M)$. By (b) M' is obtained from M by a smooth shrinking. By (a) and 14.4, (ii) of the conclusion holds.

It remains to show that $|M'|$ can be bounded in terms of L and r. For $a \in M'$ let

$$B = \bigcup\{\mathrm{cl}^{\boldsymbol{\phi}}_{M'}(\{a\}) : \boldsymbol{\phi} \in \Phi^m(M')\}.$$

Then $|B|$ is bounded in terms of $l(L)$. By 11.5 the number of conjugates of a over B is bounded in terms of $l(L)$ and r. Thus it is sufficient to bound $|\bigcup \mathscr{F}|$ for each $\mathscr{F} \in \boldsymbol{F}^m(M')$. By construction of M', $d(\mathscr{G}) = m$ for all $\mathscr{G} \in \boldsymbol{F}^m(M')$. By 11.6 $|\mathscr{F}|$ can be bounded in terms of $l(L)$, $r(M')$, and m, whence $|\mathscr{F}|$ can be bounded in terms of $l(L)$ and r. Since $|\bigcup \mathscr{F}| = |\mathscr{F}| \cdot d(\mathscr{F}) = m \cdot |\mathscr{F}|$, $|\bigcup \mathscr{F}|$ is bounded in terms of $l(L)$ and r. This completes the proof.

Intuitively, 15.1 says that every $M \in \boldsymbol{H}(L, r)$ can be "collapsed" to a structure M' isomorphic to one in a fixed finite subclass of $\boldsymbol{H}(L, r)$ such that M can be recovered from M' and the dimensions of M.

Before considering Conjecture 0.3 we prove a lemma we shall need which is of some interest in its own right.

LEMMA 15.2. *Let L be a finite relational language. Let $M_i \in \boldsymbol{H}(L)$, $\boldsymbol{\phi}_i \in \Phi(M_{i+1})$, $m_i < d_{M_{i+1}}(\boldsymbol{\phi}_i)$, and $M_i = M_{i+1}(\boldsymbol{\phi}_i, m_i)$ for all $i < \omega$. Then $M = \bigcup\{M_i : i < \omega\} \in \boldsymbol{H}(L)$. Further, if the passage from M_{i+1} to M_i is a normal one-step shrinking for all $i < \omega$, then M_0 can be obtained by shrinking M.*

PROOF. It is clear that M is homogeneous for L. Thus to show that $M \in \boldsymbol{H}(L)$ it is sufficient to show that M is stable.

If $\boldsymbol{\phi}_i = \boldsymbol{\phi}_j$ and $i < j$, then $m_i < m_j$. Therefore $\lim_{i \to \omega} m_i = \aleph_0$. From 12.2 there exists j such that $l(M_i) = l(M_j)$, $\Phi(M_i) = \Phi(M_i)$, $\approx_{M_i} = \approx_{M_j}$, $\leqq_{M_i} = \leqq_{M_j}$, and $\lim_{k \to \omega} d_{M_k}(\boldsymbol{\phi}_i) = \aleph_0$ for all $i \geqq j$. From 12.3 (vi) we can suppose that if $i \geqq j$ and $\mathscr{F}_i$, $\mathscr{F}_{i+1}$ belonging to $\boldsymbol{F}(M_i)$, $\boldsymbol{F}(M_{i+1})$ respectively are associated with $\boldsymbol{\phi} \in \Phi(M_i)$, then $\mathrm{cl}^{\boldsymbol{\phi}}_{M_{i+1}}(\{a\})$ is the image of $\mathrm{cl}^{\boldsymbol{\phi}}_{M_i}(\{a\})$ under the canonical embedding of $\bigcup \mathscr{F}_i$ into $\bigcup \mathscr{F}_{i+1}$ for all $a \in M_i$.

From 12.1 (iv) every automorphism of M_i extends to an automorphism of M_{i+1}. Therefore every automorphism of M_i extends to an automorphism of M. With each $\boldsymbol{\phi} \in \Phi(M_j)$ we can associate in an obvious way a pre-nice family $\mathscr{F}$ attached to M. For all $i \leqq j$ and $a \in M_i$, $\mathrm{cl}^{\mathscr{F}}(\{a\})$ exists and is the image of $\mathrm{cl}^{\mathscr{F}_i}(\{a\})$ under the canonical embedding of $\bigcup \mathscr{F}_i$ into $\bigcup \mathscr{F}$. Let $\Phi^{*}(M)$ denote the set of all $\phi \in \Phi(M_j)$ such that $d_M(\phi) = \lim_{i\to\omega} d_{M_i}(\phi) = \aleph_0$. Let $F^{\#}(M)$ denote the set of pre-nice families attached to M associated with pairs in $\Phi^{\#}(M)$. By choice of j, $\phi_i \in \Phi^{\#}(M)$ for all $i \geqq j$. Let $\leqq_M^{\#}$ denote the restriction of $\leqq_{M_j}$ to $\Phi^{\#}(M)$ and $\approx_M^{\#}$ the restriction of $\approx_{M_j}$. Let $\leqq^{\#}$ and $\approx^{\#}$ be the images of $\leqq_M^{\#}$ and $\approx_M^{\#}$ under the natural bijection of $\Phi^{\#}(M)$ onto $F^{\#}(M)$. It is easy to see that $\leqq^{\#}$ is transitive, $\leqq^{\#} \cap \geqq^{\#} = \approx^{\#}$, and that if $\mathcal{F}, \mathcal{G} \in F^{\#}(M)$ then $\mathcal{F} \leqq^{\#} \mathcal{G}$ iff $\mathrm{cl}^{\mathcal{G}}(\{a\}) \neq \emptyset$ for all $a \in \bigcup \mathcal{F}$.

For $a \in M$ let $B(a)$ denote $\bigcup\{\mathrm{cl}^{\mathcal{F}}(\{a\}) : \mathcal{F} \in \mathcal{F}^{\#}(M)\}$. For $a \in M_i$ let $B_i(a)$ denote $\bigcup\{\mathrm{cl}^{\phi}_{M_i}(\{a\}) : \phi \in \Phi^{\#}(M)\}$. For $a \in M$ let $C(a)$ denote the set of conjugates of a over $B(a)$ in M. For $a \in M_i$ let $C_i(a)$ denote the set of conjugates of a over $B_i(a)$ in M_i. By 11.5 $C_i(a)$ is finite. If $i \geqq j$ and $a \in M_i$, then $C_i(a) = C_{i+1}(a)$ since $\phi_i \in \Phi^{\#}(M)$. Thus, if $i \geqq j$ and $a \in M_i$, $C(a) = C_i(a)$.

Towards a contradiction suppose M is unstable. There exists $P \in S_1(\emptyset)$ such that P has Cantor–Bendixson rank ∞ in the algebra of definable subsets of M. Choose $a_0 \in M$ and maximal $B \subseteq B(a_0)$ such that C, the set of conjugates of a_0 over B in M, has CB-rank ∞. Choose $\mathcal{F}$ maximal in $F^{\#}(M)$ with respect to $\leqq^{\#}$ such that $(\bigcup \mathcal{F}) \cap (B(a_0) - B) \neq \emptyset$. Choose $I \in \mathcal{F}$ such that $(B(a_0) - B) \cap I \neq \emptyset$. Then $\mathrm{cl}^{\mathcal{G}}(\{I\}) \subseteq B$ for all $\mathcal{G} \in F^{\#}(M)$ by the maximality of $\mathcal{F}$. By 11.5 there are only a finite number of conjugates of I over B. Let

$$C_I = \{a \in C : \mathrm{cl}^{\mathcal{F}}(\{a\}) \cap I \neq \emptyset\}$$

and

$$C_I(b) = \{a \in C_I : b \in \mathrm{cl}^{\mathcal{F}}(\{a\})\} \qquad (b \in I).$$

Then C_I has CB-rank ∞, and $C_I(b)$ has CB-rank $< \infty$ for each $b \in I - B$. Since C_I has CB-rank ∞ there exist definable subsets $A(\eta) \subseteq M$ ($\eta \in {}^{<\omega}2$) such that, for all $\eta \in {}^{<\omega}2$, $A(\eta) = A(\eta^\frown\langle 0\rangle) \dot\cup A(\eta^\frown\langle 1\rangle)$ and $A(\eta) \cap C_I \neq \emptyset$. Let $D(\eta) \subseteq M$ be a finite set chosen such that $A(\eta)$ is $D(\eta)$-definable. Let $b_\eta \in I - \mathrm{cl}^I(B \cup D(\eta))$. There are two cases:

Case 1. For some $\eta \in {}^{<\omega}2$, $C_I(b_\eta) \cap A(\eta) = \emptyset$. Then $A(\eta) \cap C_I(b) \neq \emptyset$ implies $b \in \mathrm{cl}^I(B \cup D(\eta))$. Since $C_I \cap A(\eta)$ has CB-rank ∞ and $C_I = \bigcup\{C_I(b) : b \in I - B\}$ we see that

$$\bigcup\{C_I(b) : b \in \mathrm{cl}^I(B \cup D(\eta)) - B\}$$

has CB-rank ∞. Hence $C_I(b)$ has CB-rank ∞ for some $b \in \mathrm{cl}^I(B \cup D(\eta)) - B$, since a finite union has CB-rank ∞ only if one of the summands has CB-rank ∞. This contradicts the choice of B.

Case 2. Otherwise. Again we see that, if $b \in I - B$, then $C_I(b)$ has CB-rank ∞. Thus again the choice of B is contradicted.

This completes the proof that $M \in \boldsymbol{H}(L)$.

We now show that, if M_i is obtained from M_{i+1} by a normal one-step shrinking ($i < \omega$), then M_0 can be obtained by shrinking M. An L-sentence is true in M iff it is true in M_i for all sufficiently large i. By 11.7, $\Phi(M) = \Phi(M_i)$ ($i < \omega$) since by hypothesis $\Phi(M_i) = \Phi(M_0)$ ($i < \omega$). Choose j such that

$$m = \min\{d_{M_j}(\phi) : d_{M_j}(\phi) < d_M(\phi),\ \phi \in \Phi(M)\}$$

is very large compared with $l(L)$. Let

$$\Phi^{\#} = \{\phi \in \Phi(M) : d_{M_j}(\phi) < d_M(\phi)\}.$$

Let $\langle \phi_i : i < n \rangle$ be an enumeration of representatives for the equivalence classes in $\Phi^{\#}/\approx_M$ ordered such that $\phi_i \leqq_M \phi_k$ implies $k \leqq i$. Let $\mathscr{F}_i \in \boldsymbol{F}(M_j)$ be associated with ϕ_i ($i < n$). Now let $\langle N_i : i < n \rangle$ be the sequence of structures obtained by setting $N_0 = M$ and $N_{i+1} = N_i(\phi_i, d_{M_j}(\phi_i))$ ($i < n$), where the set B for this one-step shrinking is to be the image of $\bigcup \mathscr{F}_i$ under the canonical embedding of $\bigcup \mathscr{F}_i$ into $\bigcup \mathscr{G}_i$, $\mathscr{G}_i \in \boldsymbol{F}(N_i)$ being associated with ϕ_i. From 11.5 and the lemmas on shrinking in §12 it is easy to see that $N_n = M_j$. Thus M_j can be obtained by shrinking M which is clearly sufficient.

We next prove a weak form of Conjecture 0.3. Call $M' \in \boldsymbol{H}(L)$ *maximal* if there exists no $M \in \boldsymbol{H}(L)$ such that M' can be obtained from M by a normal one-step shrinking. Let $\boldsymbol{H}_m(L)$ denote the class of maximal members of $\boldsymbol{H}(L)$, and $\boldsymbol{H}_m(L, r)$ denote $\boldsymbol{H}_m(L) \cap \boldsymbol{H}(L, r)$.

THEOREM 15.3. *Let L be a finite relational language and $r < \omega$.*

(i) *$\boldsymbol{H}_m(L, r)$ contains only a finite number of isomorphism types.*

(ii) *If $M' \in \boldsymbol{H}(L, r) - \boldsymbol{H}_m(L, r)$, then there exists $M \in \boldsymbol{H}_m(L, r)$ such that M' can be obtained by shrinking M.*

PROOF. (i) Towards a contradiction suppose $\boldsymbol{H}_m(L, r)$ contains infinitely many isomorphism types. In 15.1 there are only a finite number of possibilities for M'. Hence there exists $M' \in \boldsymbol{H}(L, r)$ such that there exists $M_0, M_1, \ldots \in \boldsymbol{H}_m(L, r)$ pairwise nonisomorphic such that, for each $i < \omega$, M' satisfies the conclusion of 15.1 when $M = M_i$. We can choose $i, j < \omega$ such that $i \neq j$ and $d_{M_i} \leqq d_{M_j}$. Since $M_i \not\simeq M_j$, $d_{M_i}(\phi) < d_{M_j}(\phi)$ for some $\phi \in \Phi(M_i)$. Therefore there exists M'_i obtained by shrinking M_j such that $d_{M'_i} = d_{M_i}$ and M' can be obtained by shrinking M'_i. Since M_i satisfies the conclusion of 15.1, we have $M_i \simeq M'_i$. Therefore $M_i \notin \boldsymbol{H}_m(L, r)$. This contradiction completes the proof.

(ii) Let $M' \in \boldsymbol{H}(L,r) - \boldsymbol{H}_m(L,r)$. Choose $M_0 \in \boldsymbol{H}(L,r)$ such that M' can be obtained by shrinking M_0 and M_0 has as many of its dimensions infinite as possible. Let $\langle M_i : i < k\rangle$ with $k \leqq \omega$ be a maximal sequence in $\boldsymbol{H}(L,r)$ such that for all i, $0 < i < k$, M_{i-1} is obtained from M_i by a normal one-step shrinking. If $k < \omega$, then $M_{k-1} \in \boldsymbol{H}_m(L,r)$ and M' can be obtained by shrinking M_{k-1}. If $k = \omega$, it is clear that $M = \bigcup\{M_i : i < \omega\}$ has rank r since this can be expressed by an L-sentence. From 15.2, M_0 and hence M' can be obtained by shrinking M. But M has more infinite dimensions than M_0 contrary to the choice of M_0. Thus $k = \omega$ is impossible which completes the proof.

It is clear from 15.1 and 15.3 that Conjecture 0.1 implies Conjectures 0.2 and 0.3. Conversely, each of Conjectures 0.2 and 0.3 implies Conjecture 0.1. This is immediate in the case of 0.3. We now explain why 0.2 implies 0.1. Towards a contradiction suppose 0.2 holds and 0.1 fails for the finite relational language L. We can find a finite structure $M' \in \boldsymbol{H}(L)$ and $M_i \in \boldsymbol{H}(L)$ $(i < \omega)$ such that for all $i < \omega$ there is a smooth shrinking of M_i to M', $\mathrm{rnk}(M_i) \geqq i$, and if $N \in \boldsymbol{H}(L)$, $d_N = d_{M_i}$, and M' is obtained by shrinking N, then there is an isomorphism $\alpha : M_i \to N$ with $M' \subseteq \mathrm{fix}(\alpha)$. By thinning the sequence $\langle M_i : i < \omega\rangle$ we can suppose that, for all $i < \omega$, $d_{M_i} \leqq d_{M_{i+1}}$ and there exists $\boldsymbol{\phi} \in \Phi(M_i)$ such that $d_{M_i}(\boldsymbol{\phi}) < d_{M_{i+1}}(\boldsymbol{\phi})$. Since there is a smooth shrinking of M_{i+1} to M' there exists M'_i obtained by shrinking M_{i+1} such that $d_{M'_i} = d_{M_i}$ and M' can be obtained by shrinking M'_i. By the choice of M_i, $M'_i \simeq M_i$. Thus we can form a sequence $\langle N_i : i < \omega\rangle \in \boldsymbol{H}(L)$ such that, for each $i < \omega$, $N_i \simeq M_i$ and N_i is obtained by shrinking N_{i+1}. By 15.2, $N = \bigcup\{N_i : i < \omega\} \in \boldsymbol{H}(L)$. But, since $\mathrm{rnk}(N_i) \geqq i$ for all $i < \omega$, N is unstable. This contradiction completes the proof that Conjecture 0.2 implies Conjecture 0.1.

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